SPECIAL RELATIVITY AND CLASSICAL FIELD THEORY

— LECTURE NOTES —

PRELIMINARY VERSION

AUGUST 10, 2022

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SUMMER TERM 2020
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1. Mathematical Foundations

In this Chapter we start with providing the necessary mathematical tools. We introduce them in a way suitable for physicists, adopting the notations that are used in the Theory of Special Relativity. The summary below does not claim to be mathematically rigorous, skips many subtle details, but our aim is to understand the basic mathematical framework behind the physics of Special Relativity.

1.1. Linear algebra

1.1.1. Vector spaces

**Definition of a group**

A group is a set $G$ of elements $g \in G$ equipped with a map $\circ : G \times G \to G$, the so-called group operation, which obeys the following properties:

(a) **Associativity:** For all $a, b, c \in G$ we have $(a \circ b) \circ c = a \circ (b \circ c)$.

(b) **Identity element:** There exists an element $e \in G$ such that, for every element $a \in G$, the equation $e \circ a = a \circ e = a$ holds.

(c) **Inverse element:** For each $a \in G$, there exists an element $b \in G$ (commonly denoted by $a^{-1}$ or $-a$), such that $a \circ b = b \circ a = e$.

A group is called **commutative** or **Abelian** if $a \circ b = b \circ a$ for all $a, b \in G$.

**Remark:** In the mathematical literature, you will find various weaker definitions. For example, a semigroup is a structure where only (a) holds. If in addition (b) holds, we are dealing with a so-called monoid. Roughly speaking, a monoid is like a group, the only difference being that some elements do not have an inverse. An important example is the multiplication monoid in $\mathbb{R}$, where the element 0 does not have an inverse (since we are not allowed to divide by zero).

**Definition of a vector space**

A linear vector space over $\mathbb{R}$ is a set $V$ of vectors $v \in V$ equipped with two operations, namely, addition (‘+’) and scalar multiplication of a vector by a number (=scalar), which obey the following axioms. Let $u, v, w$ be arbitrary vectors in $V$ and let $a, b$ be scalars in $\mathbb{R}$:

(a) $V$ is a commutative group under addition.

(b) Identity element of scalar multiplication: $1v = v$. 
2 Mathematical Foundations

Figure 1.1.: Example of a vector $\mathbf{v} \in \mathbb{R}^2$ represented in two different basis systems. The origin is marked by a red bullet. The left panel shows an orthonormal basis in which the vector is represented by the components $(2,2)$. In the right panel a different basis is used, where the same vector is represented by different components $(2,3)$.

(c) Compatibility of ordinary and scalar multiplication: $a(b\mathbf{v}) = (ab)\mathbf{v}$.

(d) Distributivity with respect to vectors: $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$.

(e) Distributivity with respect to scalars: $(a + b)\mathbf{v} = a\mathbf{v} + b\mathbf{v}$.

Note that these axioms do not yet tell us what the length of a vector is, and similarly they do not yet introduce the notion of angles between vectors. To this end an additional mathematical structure is needed, namely, a scalar product (see below).

Furthermore it should be noted that this definition singles out a particular vector, namely, the zero vector $\mathbf{0}$ for which $a\mathbf{0} = \mathbf{0}$. This is the origin of the vector space.

Representation of vectors in a given basis by components

A set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\} \subset V$ is called linearly dependent if at least one of the vectors in the set can be expressed as a linear combination of the others, or equivalently, if one can find nontrivial set of coefficients $\{a^1, \ldots, a^n\}$ such that

$$
\sum_{i=1}^{n} a^i \mathbf{v}_i = 0. \tag{1.1}
$$

Otherwise, the set is called linearly independent.

A linear independent set of vectors $\{\mathbf{e}_1, \ldots, \mathbf{e}_d\} \subset V$ is called a basis of $V$ if every $\mathbf{v} \in V$ can be represented as a linear combination

$$
\mathbf{v} = \sum_{i=1}^{d} v^i \mathbf{e}_i, \tag{1.2}
$$

where $v^1, v^2, \ldots, v^d \in \mathbb{R}$ are the components of the vector and where $d$ is the dimension of the vector space.

In the Theory of Relativity we shall always use vector components with upper indices. In addition we shall use the so-called Einstein sum convention, we omit the sum symbol

---

1Physical spaces such as the 3-dimensional position space do not have an origin. For this reason, the physical position space is not a vector space but an affine space of difference vectors.

2Nontrivial means that not all of the coefficients are zero.

3For labeling the coefficients we use upper indices for reasons to become clear later.
1.1 Linear algebra

in front by automatically summing over all pairs of oppositely positioned indices in the same multiplicative term, i.e., we write

\[ \text{Einstein sum convention: } \mathbf{v} = v^i \mathbf{e}_i. \]  

(1.3)

Using a basis we represent the vector (an abstract mathematical object) in terms of components \( v^i \) (a set of numbers). It is important to note that the components depend on the choice of the basis. This means that the same vector can be represented in various different ways (see Fig. 1.1).

1.1.2. Covector spaces

From high school mathematics, you are already familiar with the concept of column and row vectors. Usually a row vector is obtained from a column vector by transposition, for example

\[
\begin{pmatrix}
1 \\
3 \\
2
\end{pmatrix}^T = \begin{pmatrix}
1 \\
3 \\
2
\end{pmatrix}
\]  

(1.4)

simply with the same components written in horizontal direction. However, as we shall see below, this holds only in the case of Euclidean vector spaces with an orthonormal basis. In Special Relativity, however, we have to deal with non-Euclidean spaces where this is no longer true, and we have to understand the difference between column and row vectors. As we shall see, column vectors represent vectors in \( V \) while row vectors represent linear functionals, also known as linear forms. The linear forms constitute a different but related vector space, the so-called covector space \( V^* \).

Linear forms

A linear form \( \alpha : V \to \mathbb{R} \) (also known as linear functional) is a linear map from the vector space \( V \) onto the scalar field \( \mathbb{R} \). It is linear with respect to both addition and scalar multiplication of vectors:

(a) Linearity under vector addition: \( \alpha(u + v) = \alpha(u) + \alpha(v) \)

(b) Linearity under scalar multiplication: \( \alpha(au) = a \alpha(u) \)

Remark: Perhaps you know this already from quantum theory. Here the ‘vectors’ are the ket vectors \( |\psi\rangle \) while the ‘linear forms’ are the bra vectors \( \langle \phi| \) . In fact, applying a bra vector to a ket vectors gives us a number, namely, \( \langle \phi| \psi \rangle \).

Representation of linear forms

Because of linearity, if we want to evaluate a linear form applied to a vector represented in a given basis, we can pull out the vector components:

\[ \alpha(u) = \alpha(u^i \mathbf{e}_i) = u^i \alpha(e_i), \]  

(1.5)
Therefore, all we need to know is how the linear form acts on the basis vectors. In other words, every linear form \( \alpha \) is fully determined by \( d \) numbers \( \alpha_i := \alpha(e_i) \), the components of the form in the dual basis. Thus, in analogy with vectors, we can simply write
\[
\alpha(u) = \alpha_i u^i, \tag{1.6}
\]
where we again use the convention of summing over oppositely oriented indices within the same multiplicative term.

**Linear forms as vectors – Covector space**

It is possible to add two linear forms simply by adding their results:
\[
\alpha + \beta : [\alpha + \beta](u) := \alpha(u) + \beta(u). \tag{1.7}
\]
The same applies to scalar multiplication, i.e., we can multiply a form by a scalar \( a \in \mathbb{R} \) in that we multiply its result:
\[
a \alpha : [a\alpha](u) := a \alpha(u). \tag{1.8}
\]
One can show that the set of linear forms equipped with addition and scalar multiplication forms a vector space in itself. This vector space is referred to as the *covector space*. Since it depends on the original vector space \( V \) it is usually denoted by \( V^* \). In fact, \( V \) and \( V^* \) are closely related. However, we have to keep in mind that both vector spaces are actually different: while \( V \) is the original vector space, \( V^* \) is the corresponding vector space of linear maps from \( V \) to \( \mathbb{R} \).

**Representation of linear forms – Covector basis \( V^* \)**

According to Eq. (1.5) a linear form \( \alpha \) is fully characterized by \( d \) numbers \( \alpha_i = \alpha(e_i) \). Conversely this means that these numbers define the linear form uniquely. This allows us to introduce a special set of \( d \) linear forms, denoted as \( \{ e^1, \ldots, e^d \} \) with upper indices, which are defined by the numbers
\[
e^i(e_j) = \delta^i_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \tag{1.9}
\]
One can easily show that \( \{ e^1, \ldots, e^d \} \) is a basis of \( V^* \). Since it depends on \( \{ e_1, \ldots, e_d \} \), the given basis of \( V \), it is called the *dual basis* or *co-basis* associated with the original basis. This associated basis allows us to be present any linear form \( \alpha \) as
\[
\alpha = \alpha_i e^i, \tag{1.10}
\]
where the coefficients \( \alpha_i \) are the *components* of the linear form. Note that the components of a linear form are written here with a *lower* indices. Applying the form to a vector \( u \) results in
\[
\alpha(u) = [\alpha_i e^i](u^i e_j) = \alpha_i u^i e^i(e_j) = \alpha_i u^i. \tag{1.11}
\]
1.1 Linear algebra

Figure 1.2.: Visualization of vectors and forms. (a) Vector space $V = \mathbb{R}^2$ with a certain non-orthogonal basis $\{e_1, e_2\}$. (b) Density plot of the basis element $e^1$ in the corresponding dual vector space $V^\ast$. Note that the corresponding gradient vector (the red arrow) is perpendicular to the basis vector $e_2$. (c) Similar representation of $e^2$. Again the corresponding gradient vector is perpendicular to $e_1$. (d) Vector $u = e_1 + 2e_2$. (e) Linear form $\alpha = e^1 - 2e^2$. (f) The linear form $\alpha$ applied to the vector $u$ gives the value $\alpha(u) = -3$.

Therefore, applying a linear form $\alpha$ to the vector $u$ amounts to pairwise multiplying and summing up their components.

Remark: This construction reminds us of a scalar product, but it is actually not a scalar product. In fact, as we will see below, a scalar product is a map $V \times V \to \mathbb{R}$ while in Eq. (1.11) we have a map $V^\ast \times V \to \mathbb{R}$.

Components of vectors carry upper indices and are called contravariant. Components of forms have lower indices and are called covariant.

Visualization of vectors and linear forms

In Fig. 1.2 we made the attempt to visualize vectors and forms in $\mathbb{R}^2$. In this figure, vectors are represented by green arrows, while linear forms (i.e. linear functions $\mathbb{R}^2 \to \mathbb{R}$) are represented as density plots with a continuous color scale. In these density plots we marked the positions, where the function returns integer values, by thin gray lines and the corresponding values by numbers in white color. In addition, we show the gradient of the linear form as a red vector.

The nontrivial interplay of vectors becomes particularly visible in a skewed (non-orthogonal) basis, as defined in panels (a). The corresponding basis elements $e^1$ and $e^2$ in the covector space $V^\ast$ are shown in panel (b) and (c), respectively. The example in the lower row of the figure illustrates that a vector $u = e_1 + 2 e_2$ applied to the form
\[ \alpha = e_1 - 2e_2 \text{ returns the value } -3. \text{ In fact, a direct calculation confirms this result:} \]

\[ \alpha(u) = [e_1 - 2e^2](e_1 + 2e_2) = e_1^4 e_1 + 2e_1^3 e_2 - 2e^2_1 e_2 - 4e^2_2 = 1 - 4 = -3. \ (1.12) \]

**Remember:** In Special Relativity we distinguish **contravariant** and **covariant** components:

- Vectors \( u \in V \) have contravariant components \( u^i \) with upper indices.
- Linear forms \( \alpha \in V^* \) have covariant components \( \alpha_i \) with lower indices.
- According to the Einstein sum convention, we automatically sum over pairs of oppositely oriented indices (up-down and down-up) within the same multiplicative term.
- Therefore, a linear form can be applied to a vector by **contracting** the indices of the corresponding components: \( \alpha(u) = \alpha_i u^i \)

### 1.1.3. Basis transformations

**Basis transformations in** \( V \)

Let \( \{e_1, \ldots, e_d\} \) be a basis in \( V \) and let \( \{\tilde{e}_1, \ldots, \tilde{e}_d\} \) be another basis in \( V \). Since every vector of the first basis can be expressed as a linear combination of the vectors in the second basis, we can always find coefficients \( A^j_i \) such that

\[ e_i = \tilde{e}_j A^j_i. \ (1.13) \]

The coefficients \( A^j_i \) can be understood as a matrix \( A \), the **transformation matrix** of the basis transformation from \( \{\tilde{e}\} \) to \( \{e\} \).

Likewise, we can express every vector of the second basis as a linear combination of vectors in the first basis:

\[ \tilde{e}_j = e_k B^k_j. \]

Inserting \( 1.14 \) into \( 1.13 \) we get \( e_i = e_k B^k_j A^j_i \), hence

\[ B^k_j A^j_i = \delta^k_i. \ (1.15) \]

This means that the transformation matrix \( B \) is the inverse of \( A \).

**Note:** The transformation matrices \( A^j_i \) and \( B^k_j \) have a lower and an upper index. Note that the **order of the indices matters**, i.e. in the present example the upper index is the first one (row index) while the lower index is the second one (column index). Only for the Kronecker symbol \( \delta_{ij} \), which represents a symmetric unit matrix, the order of the indices does not matter, which is the reason why here the indices are arranged on top of each other.

**Behavior of vector components under basis transformations**

Let us now consider a given vector \( v \in V \). This (uniquely given abstract) vector can be represented in the two basis systems as

\[ v = v^i e_i = \tilde{v}^j \tilde{e}_j, \ (1.16) \]
where the respective components $v^j$ and $\tilde{v}^i$ are generally different (see e.g. Fig. 1.1). Recall that a basis transformation is a pass"{a}{}e transforma"{c}"{a}{}sion, that is, the physical object itself (the vector $v$) remains unchanged, all what changes it the representation of the vector in terms of components.

Using Eq. (1.13) and (1.16), namely,
\[
v = v^i e_i \quad \text{(1.13)}
\]
\[
v^i A_j^i \tilde{e}_j \quad \text{(1.16)}
\]
and comparing coefficients we get the transformation law for the vector components and its inverse
\[
\begin{align*}
\tilde{v}^i &= A_j^i v^j \\
v^k &= B_k^j \tilde{v}^j .
\end{align*}
\] (1.18)

**Basis transformations in $V^*$**

Similarly we can show that the dual basis transforms inversely under basis transformations, i.e.
\[
e^k = B^k_j \tilde{e}^j , \quad \tilde{e}^i = A^i_j e^j .
\] (1.19)

This can be proven by verifying the definition property (1.9). In fact, we find that:
\[
e^k(e_i) = B^k_j A^j_i \tilde{e}^j(e_i) = B^k_j \delta^k_i \]
\[
\tilde{e}^i(\tilde{e}_j) = A^i_j B^k_j \tilde{e}^k(e_k) = A^i_j \delta^i_j .
\] (1.20)

Consider a given linear form $\alpha \in V^*$. This linear form can be represented in these two basis systems with different components:
\[
\alpha = \alpha_i e^i = \tilde{\alpha}_i \tilde{e}^i .
\] (1.22)

Since $\alpha = \tilde{\alpha}_i \tilde{e}^i = \tilde{\alpha}_i A^j_i e^j = \alpha_j e^j$, we can derive the transformation law for the components and its inverse:
\[
\begin{align*}
\alpha_j &= \tilde{\alpha}_i A^i_j \\
\tilde{\alpha}_i &= \alpha_j B^i_j
\end{align*}
\] (1.23)

**Note:** It is instructive to compare the transformation laws (1.18) and (1.23). The vector components are contracted with the right index of the respective matrix, that is, the matrix act to the right, like on ket vectors in quantum mechanics. Contrarily, the components of the forms are contracted with the first index of the matrix, hence they behave like bra vectors in quantum mechanics. In fact, in the quantum formalism, ket vectors are elements of $V$ while bra vectors are elements of $V^*$. 

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1.1.4. Tensors

**Definition**

A tensor $T$ of rank $(p,q)$ is a multilinear map that maps $p$ linear forms $\alpha^{(1)}, \ldots, \alpha^{(p)}$ and $q$ vectors $v_{(1)}, \ldots, v_{(q)}$ onto a scalar in such a way that the map is linear in each argument:

$$\alpha^{(1)}, \ldots, \alpha^{(p)}, v_{(1)}, \ldots, v_{(q)} \mapsto T(\alpha^{(1)}, \ldots, \alpha^{(p)}, v_{(1)}, \ldots, v_{(q)}) \in \mathbb{R} \quad (1.24)$$

This concept includes the following special cases:

- Tensors of rank $(0,0)$ are scalars.
- Tensors of rank $(1,0)$ are vectors.
- Tensors of rank $(0,1)$ are linear forms.
- Tensors of rank $(1,1)$ are linear maps.

Like vectors, tensors are abstract mathematical objects.

**Representation of tensors**

In a given basis of $V$ and $V^*$ the action of the tensor can be represented by

$$T(\alpha^{(1)}, \ldots, \alpha^{(p)}, v_{(1)}, \ldots, v_{(q)}) = T_{i_1, \ldots, i_p, j_1, \ldots, j_q} \alpha^{i_1} \cdots \alpha^{i_p} v_{j_1} \cdots v_{j_q} \quad (1.25)$$

where

$$T_{i_1, \ldots, i_p, j_1, \ldots, j_q} = T(e^{i_1}, \ldots, e^{i_p}, e^j_{j_1}, \ldots, e^j_{j_q}) \quad (1.26)$$

are the components of the tensor. As can be seen, the tensor components carry $p$ contravariant (upper) and $q$ covariant (lower) indices.

Under a basis transformation each index gives rise to an individual transformation matrix. As a rule of thumb, the contravariant (upper) indices transform like vector indices while the covariant (lower) indices transform like an index of a linear form. That is, each index needs its own transformation matrix:

$$T^{k_1, \ldots, k_p}_{l_1, \ldots, l_q} = A^{k_1}_{l_1} \cdots A^{k_p}_{l_p} T^{i_1, \ldots, i_p}_{j_1, \ldots, j_q} B^{i_1}_{j_1} \cdots B^{i_p}_{j_p} \quad (1.27)$$

**Remember:** A tensor is a multilinear map with components that transform like vectors in the upper indices and like forms in the lower indices.

1.1.5. Scalar product

In what we have encountered so far, there is nothing related to distances and angles. In order to incorporate these notions, we have to introduce an additional mathematical structure, namely, the *scalar product*, also known as the *dot product*. As we will see, this product tightly connects $V$ and $V^*$, allowing us to switch freely between the two spaces.
Definition

A scalar product is a symmetric bilinear positive-definite map \( \langle \cdot, \cdot \rangle : V \otimes V \to \mathbb{R} \). More specifically, if \( \mathbf{u}, \mathbf{v}, \mathbf{w} \in V \) and \( a, b \in \mathbb{R} \), a scalar product is defined by the following properties:

- Bilinearity: \( \langle a \mathbf{u} + b \mathbf{v}, \mathbf{w} \rangle = a \langle \mathbf{u}, \mathbf{w} \rangle + b \langle \mathbf{v}, \mathbf{w} \rangle \)
- Symmetry: \( \langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle \).
- Positive definiteness: \( \langle \mathbf{v}, \mathbf{v} \rangle \geq 0 \) and \( \langle \mathbf{v}, \mathbf{v} \rangle = 0 \Leftrightarrow \mathbf{v} = 0 \)

In many textbooks the scalar product is also denoted by \( g \), using the notation

\[
g(\mathbf{u}, \mathbf{v}) := \langle \mathbf{u}, \mathbf{v} \rangle
g (1.28)
\]

This is in fact the standard notation used in the Theory of Relativity.

Representation of the scalar product in a given basis: The metric tensor

Since the scalar product is a bilinear function that maps two vectors onto a number, it can be understood as a tensor of rank (0,2). As such, it can be represented in a given basis by a matrix \( g_{ij} \) with two lower indices, the so-called metric tensor, which is defined by

\[
g_{ij} := g(\mathbf{e}_i, \mathbf{e}_j)
g (1.29)
\]

Using this matrix representation and the usual representation of the two vectors on which it acts, namely, \( \mathbf{u} = u^i \mathbf{e}_i \) and \( \mathbf{v} = v^j \mathbf{e}_j \), the scalar product can be expressed in terms of the components by:

\[
g(\mathbf{u}, \mathbf{v}) = g_{ij} u^i v^j.
g (1.30)
\]

The matrix \( g_{ij} \) has the following properties:

- All entries are real-valued.
- Because of the symmetry the matrix is symmetric (invariant under transposition).
- Positive definiteness means that all eigenvalues of \( g_{ij} \) are positive.

Behavior of the metric tensor under basis transformations

While the scalar product \( g \) is an abstract mathematical object independent of the chosen basis, reflecting a physical reality, its representation in terms of a matrix \( g_{ij} \) depends on the choice of the basis. In particular, the matrix is expected to change under basis transformations. According to Eq. (1.27) the matrix \( g_{ij} \) is expected to change as

\[
\tilde{g}_{ij} = g_{ij} B^i_k B^j_l.
g (1.31)
\]

where \( g_{ij} = g(\mathbf{e}_i, \mathbf{e}_j) \) and \( \tilde{g}_{ij} = g(\tilde{\mathbf{e}}_i, \tilde{\mathbf{e}}_j) \). This transformation behavior ensures that the result of the scalar product is indeed a scalar, i.e., a quantity which is invariant under
basis transformations:
\[ g(u, v) = g_{ij}u^i v^j = \tilde{g}_{kl} \tilde{u}^k \tilde{v}^l. \] (1.32)

**Proof:** To prove this invariance, recall that the transformation matrices \( A \) and \( B \) are inverse to each other. Inserting the transformation laws they drop out:
\[ \tilde{g}_{kl} \tilde{u}^k \tilde{v}^l = g_{ij} B^i_k B^j_l A^k_m A^l_n u^m v^n \]
\[ = \delta_{ij} \]
\[ = g_{ij} u^i v^j. \] □

**Induced norm and distance**

In Mathematics it is well known that a scalar product induces two important structures:

- The **norm** \( || \cdot || : V \to \mathbb{R} \) which returns the length of a vector, defined by
  \[ ||u|| = \sqrt{g(u, u)} \] (1.33)

- The **metric** \( d : V \times V \to \mathbb{R} \), returning the distance between two vectors, defined by
  \[ d(u, v) = ||u - v||. \] (1.34)

Loosely speaking, the scalar product turns the linear vector space, which does not incorporate the notion of lengths and distances, into a metric space. This explains why in the Theory of Special Relativity the tensor \( g_{ij} \) is often referred to as the **metric tensor** or even shorter as the **metric**.

**Remark:** Note that there is a slightly different language used by mathematicians and physicists. For mathematicians is a measure \( d(u, v) \) with certain defining properties. Physicists and researchers in relativity usually refer to the metric as the scalar product, represented by the metric tensor, which is actually stronger than the ‘mathematical’ metric.

**Euclidean metric**

The best known example of a metric is the **Euclidean metric** in \( \mathbb{R}^d \) which in the canonical standard basis is given by
\[ g_{ij} = g(e_i, e_j) = \delta_{ij}. \] (1.35)

Obviously the Euclidean metric is symmetric and positive definite. Using Eq. (1.30) the scalar product can be written as
\[ \langle u, v \rangle = g(u, v) = g_{ij}u^i v^j = \sum_{i=1}^{d} u^i v^i. \] (1.36)

This scalar product induces the usual Euclidean norm (the Pythagorean theorem)
\[ ||v|| = \sqrt{\sum_{i=0}^{d} (v^i)^2}. \] (1.37)
There is only one vector with zero length, namely, the zero vector.

**Pseudometric and signature**

A pseudometric space is a vector space equipped with a scalar product that is no longer positive definite. The most important property of a pseudometric space is that the distance between two different points can be zero, or equally, nonzero vectors can have the length zero. As we shall see below, the 3+1-dimensional space-time of Special Relativity is in fact such a pseudometric space.

Example: The simplest example is a two-dimensional vector space equipped with the scalar product $g(u, v) = u^1 v^1 - u^2 v^2$. If we are interested in finding vectors $v$ with length zero, i.e., $g(v, v) = 0$ we get $(v^2)^2 = (v^1)^2 \Rightarrow v^2 = \pm v^1$. This is the so-called light cone of the metric, which will play an important role in the Theory of Relativity (highlighted by red lines in the figure below).

Such a non-positive-definite scalar product turns out to have a metric tensor $g_{ij}$ with not only positive but also negative eigenvalues. The string of the signs of these eigenvalues, sorted either by $+ \ldots + \ldots -$ or $- \ldots - + \ldots +$, is called as the signature of the metric. An ordinary positive-definite metric has the signature $+ \ldots +$, while a pseudo metric has a mixed signature.

In anticipation of the following, it should be noted that the Special Theory of Relativity is based on a metric with the signature $- + ++$.

### 1.1.6. Mapping vectors to forms and vice versa

**Mapping vectors to forms**

The scalar product can be used to associate with each vector $u \in V$ a unique linear form $\alpha \in V^*$ as follows:

Recall that $g(\cdot, \cdot)$ is a bilinear symmetric map $V \times V \to \mathbb{R}$. Hence, if we plug a given vector $u \in V$ into one of its slots (because of symmetry it does not matter which one, say the second one), then the resulting map $g(\cdot, u)$ can be understood as a linear map $V \to \mathbb{R}$, and this is, by definition, nothing but a linear form:

$$u \in V \mapsto \alpha \in V^*: \quad \alpha(v) := g(v, u). \quad (1.38)$$

**Lowering indices**

As any linear form, the associated form $\alpha$ can be represented in the dual basis by $\alpha = \alpha_i e^i$ with certain components $\alpha_i$. How are these components related to those of the original vector $u = u^i e_i$? In order to find this relation, we can use the important relation (1.9), namely,

$$e^i(e_j) = \delta^i_j. \quad (1.39)$$
Hence, with the representations
\[
\alpha(v) = \alpha_i v^i e^i(e_j) = \alpha_i v^j \delta^i_j = \alpha_i v^i \\
g(v, u) = g(v^i e_i, u^j e_j) = v^i u^j g(e_i, e_j) = g_{ij} u^i v^j
\]  
we get \(\alpha_i v^i = g_{ij} u^j\), and since this relation has to hold for all \(v \in V\), we arrive at \(\alpha_i = g_{ij} u^j\). As the form \(\alpha\) is associated with the vector \(u\), we may simply use the same character and denote its components by \(u_i\) instead of \(\alpha_i\), giving the important relation
\[
(1.41)
\]
This is the well-known rule for lowering the index of the vector components \(u^j\) by applying the matrix \(g_{ij}\). It is similar to converting a column vector to a row vector, or likewise, a ket vector to a bra vector.

**Note:** It is important that this lowering an index must not be used on the basis elements themselves, i.e. we are not allowed to write \(e_i = g_{ij} e^j\). In fact, this equation does not make any sense since \(e_i\) is an element of \(V\) and \(g_{ij} e^j\) is an element of the dual space \(V^*\).

### Riesz representation theorem

So far we have shown that each vector \(u \in V\) can be associated with a unique linear form \(\alpha \in V^*\). But what happens in opposite direction? Here we can use the Riesz representation theorem that you may have encountered in your course on linear algebra:

**Recall:** Riesz representation theorem: Let \(V\) be a vector space equipped with a scalar product \(g\). Then for every linear form (functional) \(\alpha \in V^*\) there is a unique vector \(u \in V\) such that \(\alpha(v) = g(v, u)\) for all \(v \in V\).

The Riesz representation theorem basically tells us that the map \(V \to V^* : u \mapsto \alpha\) introduced above is invertible, that is, we are dealing here with an isomorphism. In particular, any form \(\alpha \in V^*\) can be mapped uniquely to a certain vector \(u \in V\).

### Induced scalar product on \(V^*\)

Consider two linear forms \(\alpha\) and \(\beta\) which by virtue of Riesz’s theorem correspond to the vectors \(u\) and \(v\), respectively. This correspondence allows us to define a scalar product \(g^*\) on \(V^*\) simply by defining
\[
g^*(\alpha, \beta) := g(u, v).
\]  
(1.42)
It is in fact easy to check that \(g^*(\cdot, \cdot)\) meets all the requirements of a scalar product, in particular symmetry, bilinearity, and (if required) positive definiteness. Therefore, if \(V\) is equipped with a scalar product \(g\), it is automatically equipped with a corresponding scalar product \(g^*\) on \(V^*\).

In a given basis the induced scalar product can be computed exactly in the same way as the ordinary one in Eqs. (1.29)-(1.30). To this end we define the **contravariant metric**
\[
g^{ij} := g^*(e^i, e^j),
\]  
(1.43)
where we omitted the star, writing \( g^{ij} \) instead of \( g^{*ij} \), because the upper position of the indices already tells us that we are dealing with the induced metric on \( V^* \). With this definition the induced scalar product is given by
\[
g^*(\alpha, \beta) = g^{ij} \alpha_i \beta_j.
\] (1.44)

It turns out that the matrices \( g^{ij} \) and \( g_{ij} \) are inverse to each other:
\[
\begin{bmatrix} g^{ij} \end{bmatrix} = \begin{bmatrix} g_{ij} \end{bmatrix}^{-1} \iff g^{ij} g_{jl} = \delta^i_l
\] (1.45)

**Proof:** To prove these relations, we use the representations in terms of components
\[
u = u^i e_i, \quad v = v^j e_j, \quad \alpha = u^k e_k, \quad \beta = v^l e_l,
\]
where \( \alpha_k \equiv u^k \) and \( \beta_l \equiv u^l \). Inserting these representations into Eq. (1.42) we get:
\[
l.h.s.: \quad g^*(\alpha, \beta) = \sum_{i=1}^n \sum_{j=1}^n \alpha_i \beta_j g^{ij} = \sum_{i=1}^n \sum_{j=1}^n \alpha_i \beta_j g^{ij} u^i v^j = g^{ij} u^i v^j \]
\[
r.h.s.: \quad g(u, v) = g_{kl} u^k v^l.
\]
Since both sides have to coincide for all \( u, v \in V \), we can conclude that \( g^{ij} g_{ik} = g_{kl} \). Using the symmetry \( g_{ik} = g_{ki} \) we get
\[
g_{kl} g^{ij} = g_{kl} \Rightarrow g^{ij} g_{jl} = \delta^i_l.
\] (1.47)

**Behavior of the dual metric tensor under basis transformations**

Since \( g^{ij} \) is a contravariant tensor of rank \((2,0)\), Eq. (1.27) tells us that it changes under basis transformations as
\[
g^{kl} = A^k_i A^j_l g^{ij}
\] (1.46)
where \( g^{ij} = g(e^i, e^j) \) and \( \tilde{g}^{ij} = g(\tilde{e}^i, \tilde{e}^j) \). Note that this tensor transform inversely to the one in Eq. (1.31). In particular, the relation (1.45) is preserved:
\[
g^{ij} g_{jl} = \tilde{g}^{ij} \tilde{g}_{jl} = \delta^i_l.
\] (1.47)

**Raising indices**

Having established the map in opposite direction, we are now led to the law for *raising the index* of a component:
\[
u^i = g^{ij} u_k.
\] (1.48)

The can be proven in the same way as Eq. (1.41), but as we have shown that \( g_{ij} \) and \( g^{jk} \) are inverse to each other, it is obvious that (1.48) is the inverse of (1.41) since
\[
u_i \stackrel{1.41}{=} g_{ij} u^j \stackrel{1.48}{=} g_{ij} \delta^k g^{jk} u_k \stackrel{1.45}{=} \delta^k u_k = u_i.
\] (1.49)
Remark: In mathematics, the map $u \leftrightarrow \alpha$ is known as the **musical isomorphism**. The map $V^* \rightarrow V$ is denoted by the symbol $\sharp$, while the inverse map $V \rightarrow V^*$ is denoted by the symbol $\flat$.

**About the purpose of upper and lower indices**

The use of upper and lower indices in the Theory of Special Relativity is useful in the following sense:

- It helps us to distinguish vectors with contravariant components and functionals (linear forms) with covariant components.
- In practice we can freely switch between upper and lower indices by applying the raising and lowering operations with the respective metric tensors.
- The possibility of lowering and raising combined with the Einstein convention to sum over oppositely oriented indices in the same term allows us to write the scalar product as
  \[
  g(u, v) = u^i v_i = u_i v^i. \tag{1.50}
  \]

As you will see below, the Theory of Special Relativity makes extensive use of lower and upper indices. However, this is primarily a convenient way to write formulas in a compact form. We could write down all relations in terms of vector components only (using only objects with upper indices), but then we would have to insert $g_{ij}$’s in many places. Therefore, the formalism of upper and lower indices is introduced mainly as a convenient and compact notation.

<table>
<thead>
<tr>
<th>Object</th>
<th>Formula</th>
</tr>
</thead>
<tbody>
<tr>
<td>Basis of $V$</td>
<td>${e_1, e_2, \ldots}$</td>
</tr>
<tr>
<td>Dual basis of $V^*$</td>
<td>${e^1, e^2, \ldots}$, $e^i(e_j) = \delta^i_j$</td>
</tr>
<tr>
<td>Vector representation</td>
<td>$v = v^i e_i$</td>
</tr>
<tr>
<td>Linear form representation</td>
<td>$\alpha = \alpha_j e^j$</td>
</tr>
<tr>
<td>Metric tensor</td>
<td>$g_{ij} = g(e_i, e_j)$</td>
</tr>
<tr>
<td>Inverse metric tensor</td>
<td>$g^{jk} = g^*(e^j, e^k)$</td>
</tr>
<tr>
<td>Lowering indices</td>
<td>$u_i = g_{ij} u^j$</td>
</tr>
<tr>
<td>Raising indices</td>
<td>$T_{ij} = g_{kl} g^{jk} T^{kl}$</td>
</tr>
<tr>
<td>Scalar product</td>
<td>$\langle u, v \rangle = g(u, v) = g_{ij} u^i v^j = g^{ij} u_i v_j = u^i v^j = u_i v^j$</td>
</tr>
<tr>
<td>Basis transformation</td>
<td>$\tilde{e}_j = e_j B^k_j$, $e_i = \tilde{e}_j A^j_i$, $B^k_j A^j_i = \delta^k_i$</td>
</tr>
<tr>
<td>Transformation of vectors</td>
<td>$\tilde{\alpha}_i = A^j_i \alpha^j$, $\nu^k = B^k_j \nu^j$</td>
</tr>
<tr>
<td>Transformation of forms</td>
<td>$\tilde{\alpha}_k = \alpha_k B^1_k$, $\alpha_k = \tilde{\alpha}_1 A^1_k$</td>
</tr>
<tr>
<td>Transformation of metric</td>
<td>$\tilde{g}<em>{kl} = g</em>{ij} B^i_k B^j_l$, $\tilde{g}_{ij} = A^k_i A^l_j g^{kl}$</td>
</tr>
</tbody>
</table>
What you should never do:

Never raise or lower the index of a basis vector like this:

\[ e^i \rightarrow g^{ij} e_j \]

The reason is that \( e_j \in V \) while \( e^i \in V^* \). This does not make sense. It is like comparing apples with bananas.
1.2. Symmetries of metric spaces

1.2.1. Coordinate systems

**Coordinates**

Consider a \(d\)-dimensional physical space. In full generality, this could be even a curved space such as a sphere, but here let us for simplicity think of a flat vector space \(V = \mathbb{R}^d\), where each physical position is represented by a position vector \(v \in V\).

A **coordinate system** is a set of \(d\) functions \(x^i : V \to \mathbb{R}\) labeled by \(i = 1, \ldots, d\). It maps a given point \(v \in V\) to \(d\) real numbers \(x^1(v), \ldots, x^d(v)\). The purpose of a coordinate system is that the values of the coordinates tell us uniquely where to find the point, just like latitude and longitude uniquely specify each point on earth. This means that the coordinate mapping has to be invertible:

\[
\text{position } v \in V \iff \text{coordinates } \{x^1(v), \ldots, x^d(v)\}.
\]

In principle the coordinate functions \(x^i\) could be any smooth nonlinear functions, so they must not be confused with components of linear forms. For example, the well-known polar coordinates are non-linear. The only requirement is the invertibility.\(^4\)

Coordinate systems are often visualized by **coordinate surfaces** which are defined as \(d-1\)-dimensional hyperplanes where one of the coordinates \(x^i\) is kept constant. In the special case of \(d = 2\) dimensions, these coordinate surfaces are just the usual **grid lines** of the coordinate system, as visualized in Fig. 1.3. This figure shows three example, namely

- (a) an arbitrary curved nonlinear coordinate system,
- (b) a nonlinear coordinate system, namely, the well-known polar coordinates, and
- (c) a Cartesian curvilinear system, where the grid lines are straight but not necessarily orthogonal.

\(^4\)Without going into detail here, the requirement of invertibility usually means that that the Jacobi determinant does not vanish.
**Cartesian curvilinear coordinate systems**

Similar to the special case (c) shown in the figure, the flat space-time in the Theory of Special Relativity is usually described by *curvilinear coordinate systems*. This means that the “physical” space-time is modeled by a vector space $V$ whose positions are represented by vectors $v \in V$. Recall that in a curvilinear coordinate system the coordinate functions $x^i(v)$ are linear, and therefore the coordinate surfaces are straight lines in two dimensions and flat hypersurfaces in higher dimensions, respectively.

If the coordinate functions are linear, then the inverse coordinate map $v(x^1, \ldots, x^d)$ will be linear as well, i.e., it can be written as

$$v(x^1, \ldots, x^d) = x^i e_i + a. \quad (1.51)$$

Here $a$ is the position where the origin of the coordinate system is located, while the vectors $e_i$ are given by the partial derivatives

$$e_i := \frac{\partial v}{\partial x^i}. \quad (1.52)$$

Since the coordinate map is invertible, this set of vectors is linear independent, hence it constitutes a particular basis of $V$. This special basis is referred to as the *coordinate basis* of the coordinate system.

Note that we defined the basis vectors with lower indices as partial derivatives with respect to something with upper indices. This structure is ubiquitous in Special and General Relativity: Derivatives with respect to something contravariant are covariant and vice versa. We will shortly understand why.

**Remember:** The coordinate basis in curvilinear coordinates is a special basis defined by $e_i := \frac{\partial v}{\partial x^i}$. In this basis the vector components of $v - a$ coincide with the coordinates. In most cases the origin of the coordinate system is chosen to coincide with the origin of the vector space, setting $a = 0$.

### 1.2.2. Transformations

**Active and passive transformations**

In physics, groups play an important role in the context of *transformations*. There are two types of transformations:

- **Active** transformations are operations that modify (transform) a physical object in a reversible manner. For example, if we move the red bullet by the distance $a$ to the right, it changes its physical location and consequently its coordinate changes by $x \mapsto \tilde{x} = x + a$:

  [Diagram showing a bullet moving from $x$ to $\tilde{x}$]

- **Passive** transformations are operations that change the *representation* of a physical object. For example, we could move the coordinate system to the left, leaving the
actual physical object in its place:

\[ x \rightarrow \tilde{x} = x + a, \]

As a result, the coordinate of the object changes by \( x \rightarrow \tilde{x} = x + a \), that is, it changes exactly in the same way as for active transformations. However, the object as such remains in place.

The reason why both transformations, which are so different in nature, are described by the same formula is very simple: A coordinate only specifies the relative distance between the origin of the coordinate system (the reference frame) and the object. Therefore, we cannot distinguish whether the object is displaced to the right or the reference frame is shifted to the left. The same applies to other types of transformations, e.g., rotations and translations in time.

**Transformation of functions**

Instead of a physical object located at the position \( x \), we could also consider a function, as shown in Fig. 1.4. If we move a given function \( f \) to the right by the distance \( a \), we obtain a different function \( \tilde{f} \). As can be seen, this leads us to the identity \( \tilde{f}(\tilde{x}) = f(x) = f(\tilde{x} - a) \). In the sloppy notation of physicists such an active transformation is usually denoted as a replacement \( f \rightarrow \tilde{f} \) with

\[
f(x) \rightarrow \tilde{f}(x) = f(x - a)
\]

or even shorter as \( f(x) \rightarrow f(x - a) \), where we dropped the tilde above the \( x \). Note that we have \( x - a \) on the right hand side. That is, moving a function to the right amounts to moving its argument to the left. Or more generally:

**If we want to apply an active transformation to a function we have to apply the inverse transformation to its argument.**

The same holds for passive transformations of a function, as sketched on the right hand side of Fig. 1.4. If we change the reference frame, for example, by moving the coordinate system to the left by the distance \( a \), we have to replace \( f(x) \rightarrow \tilde{f}(x - a) \). This highlights again the equivalence of active and passive transformations.
Coordinate transformations

A coordinate transformation is a transformation that maps the coordinates \( \{x^1, \ldots, x^d\} \) to \( \{\tilde{x}^1, \ldots, \tilde{x}^d\} \). This map could be linear or non-linear. The map has to be locally invertible, meaning that the Jacobi determinant \( |\partial \tilde{x}^i / \partial x^j| \) has to be nonzero.

As a special case, we will consider linear coordinate transformations of the form

\[
\tilde{x}^i = \Lambda^i_j x^j + s^i. \tag{1.54}
\]

Here \( s^i \) describes a translation while \( \Lambda^i_j \) is a transformation matrix with non-vanishing determinant. Note that we can express this matrix as

\[
\Lambda^i_j = \frac{\partial \tilde{x}^i}{\partial x^j}. \tag{1.55}
\]

A coordinate transformation can either be interpreted as an active transformation that manipulates physical objects, or as a passive transformation that modifies the coordinate representation. In the passive case the transformation matrix \( \Lambda^i_j \) is just identical with the matrix \( A^i_j \) in the preceding section.

Transformation behavior of partial derivatives

A frequently encountered problem with coordinate transformations is the following. Suppose that we transform the coordinates according to Eq. (1.54). How do the corresponding partial derivatives transform?

To answer this question, it is important to realize that partial derivatives applied to the coordinates obey a certain algebra, namely

\[
\frac{\partial}{\partial x^k} x^i = \delta^i_k. \tag{1.56}
\]

**Remark:** An algebra (an Arabic terminus) generally consists of

- a set of symbols, say \( A, B, C \),
- words formed from symbols, like \( ABABA \),
- a space of linear combinations of words, such as \( 2AB + 7BA \),
- certain algebraic rules (commutation relations) restricting the size of the word space.

In our case the symbols are \( x^i \) and \( \partial / \partial x^k \), the algebraic rules are the relations (1.56).

Obviously, the partial derivatives with respect to the new coordinates should obey the same algebra with the new coordinates, that is

\[
\frac{\partial}{\partial \tilde{x}^k} \tilde{x}^i = \delta^i_k. \tag{1.57}
\]

One can easily show that this is only possible if the partial derivatives transform inversely, that is, they have to transform like linear forms:

\[
\frac{\partial}{\partial \tilde{x}^k} = \frac{\partial}{\partial x^l} (\Lambda^{-1})^l_k \tag{1.58}
\]
Proof: To prove this transformation behavior, we start with the assumption that the partial derivatives with respect to the new coordinates can be expressed as a linear combination of the old partial derivatives
\[ \frac{\partial}{\partial \tilde{x}^k} = \sum_l \frac{\partial}{\partial x^l} B^l_k \]
with some coefficients \( B^l_k \). Then we apply this to \( \tilde{x}^i \) in (1.54):
\[ \delta^i_k \frac{\partial}{\partial \tilde{x}^k} \tilde{x}^i = \sum_l \frac{\partial}{\partial x^l} B^l_k \left( \Lambda^i_j x^j + \delta^i \right) = \sum_l \Lambda^i_j B^l_k \frac{\partial}{\partial x^j} x^i = \Lambda^i_j B^i_j \]
This means that \( 1 = \Lambda B \), hence \( B \) is the inverse of \( \Lambda \).

Since partial derivatives transform covariantly like linear forms, it is convenient and meaningful to define a shortcut notation with a covariant (lower) index:
\[ \partial_i := \frac{\partial}{\partial x^i} \] (1.59)
and likewise in the tilde coordinates
\[ \tilde{\partial}_i := \frac{\partial}{\partial \tilde{x}^i} \] (1.60)
both obeying the algebra
\[ \partial_i x^j = \delta_i^j \tilde{x}^j = \delta_i^j. \] (1.61)
Similarly, we can define partial derivatives with upper with respect to \( x^k \):
\[ \partial^i := \frac{\partial}{\partial x^i} \] (1.62)
and, consistently, we can obtain this derivative by raising the index:
\[ \partial^i = g^{ij} \partial_j \] (1.63)

The partial derivative with respect to a contravariant (upper index) component transforms like a component with a lower index, and vice versa.

1.2.3. Lie groups and Lie algebras

Roughly speaking, Lie groups are groups with group elements which can be Taylor-expanded around the identity. The group elements of Lie groups (usually transformations) are typically controlled by one or several continuous parameters. For example, the group of rotations in 2D is a Lie group, controlled by a single parameter, namely, the rotation angle. The parameters allow us to create elements arbitrarily close to the identity, in Physics referred to as infinitesimal transformations. As we will see below, infinitesimal transformations are generated by a set of generators. These generators and their commutation relations constitute the Lie algebra of the Lie group. Thus, the Lie algebra describes the local properties of the Lie group in the vicinity of the identity.
The translation group

The simplest example of a Lie group is the translation group $T$. Its elements $T_a \in T$ translate an object by the vector $a$. The neutral element is $T_0 = 1$ and the inverse element of $T_a$ is $T_{-a}$. The group operation is sequential execution, and it is clear that $T_a \circ T_b = T_{a+b}$. Hence the translation group is isomorphic to vector addition and thus it is an Abelian (commutative) group.

For simplicity, let us have a look at the one-dimensional case and let us consider an infinitesimal translation $T_\epsilon$. This translation shifts the function $f(x)$ to the right by the infinitesimal distance $\epsilon \ll 1$, as shown in the figure. As we have seen in Eq. (1.53), the shifted function drawn in red color is given by $\tilde{f}(x) = f(x-\epsilon)$. Since $\epsilon \ll 1$ this can be Taylor-expanded:

$$\tilde{f}(x) = f(x-\epsilon) \approx f(x) - \epsilon f'(x)$$ (1.64)

or equivalently

$$\tilde{f}(x) \approx \left(1 - \epsilon \frac{d}{dx}\right) f(x).$$ (1.65)

Here we can see that the derivative $\frac{d}{dx}$ generates the infinitesimal translation. In other words, the derivative operator $\frac{d}{dx}$ is the so-called generator of the translation group.

If we want to translate the function by a finite distance $a$, we could divide the finite shift into infinitely many ($N \to \infty$) infinitesimal shifts:

$$\tilde{f}(x) = f(x-a) = \lim_{N \to \infty} \left(1 - a \frac{d}{N \, dx}\right)^N f(x).$$ (1.66)

Does this formula remind you of something? Right! This is nothing but the well-known representation of the exponential function

$$e^x = \lim_{N \to \infty} \left(1 + \frac{x}{N}\right)^N$$ (1.67)

Hence

$$\tilde{f}(x) = \exp\left(-a \frac{d}{dx}\right) f(x).$$ (1.68)

Here we see that the exponential function applied to the generator (times the translation distance) is a machine that carries out the translation, or in scientific parlance, is a representation of the group element.

In higher dimensions, we can carry out a translation by a finite vector $a$ exactly in the same way:

$$\tilde{f}(x) = e^{-a \cdot \nabla} f(x) = e^{-a \cdot \partial_i} f(x).$$ (1.69)
Finite transformation = \( \exp(-\text{distance} \times \text{generator}) \)

**The orthogonal group**

Consider the Euclidean \( \mathbb{R}^2 \) with a standard Cartesian basis \( e_1, e_2 \). The group of rotations, the so-called *special orthogonal group* \( \text{SO}(2) \), consists of elements \( R_\phi \in \text{SO}(2) \) with the well-known matrix representation

\[
\begin{pmatrix}
\tilde{x}^1 \\
\tilde{x}^2
\end{pmatrix} =
\begin{pmatrix}
\cos \phi & -\sin \phi \\
\sin \phi & \cos \phi
\end{pmatrix}
\begin{pmatrix}
x^1 \\
x^2
\end{pmatrix}
\tag{1.70}
\]

The group of rotations \( \text{SO}(2) \) is a commutative Lie group with a single parameter \( \phi \) and a single generator \( L \) with the following representation (see exercises):

\[
R_\phi = \exp(\phi L), \quad L = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}
\tag{1.71}
\]

Rotations are special in so far as they preserve the Euclidean distance between points, that is, they are *isometries* in the \( \mathbb{R}^2 \) with the Euclidean standard metric.

In higher dimensions the situation is more involved since the orientation of the rotation axis begins to matter. For example, in three dimensions, we have three generators

\[
L_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad L_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad L_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}
\tag{1.72}
\]

and the corresponding group elements of the group of rotations in three dimensions \( \text{O}(3) \) are given by

\[
R_\vec{\phi} = \exp(\vec{\phi}^i L_i),
\tag{1.73}
\]

where the *direction* of the vector \( \vec{\phi} \) defines the rotation axis, the *orientation* of the vector the sense of the rotation (left, right), and the *length* of the vector the rotation angle. It is important to note that the three generators do not commute:

\[
[L_i, L_j] = \epsilon_{ijk} L_k.
\tag{1.74}
\]

This means that the group \( \text{O}(3) \) is a non-commutative group. The set of generators and their commutation relations constitute the so-called *Lie algebra* of the group. Roughly speaking, the Lie algebra tells us how the group behaves locally in a Taylor expansion to first order.

**Remark:** Rotations in \( \text{SO}(n) \) are not the only distance-preserving maps. Another important discrete symmetry group is the group of reflections at hyperplanes. Rotations and reflections together form the so-called *orthogonal group* \( \text{O}(n) \). If we were to include translations as well, which are also distance-preserving, we obtain the *Euclidean group* \( \text{E}(n) \).
Note: Sometimes, people prefer to write \( R_\varphi = \exp(-i\varphi L_i) \) with a complex \(-i\) inside the exponential. Then the generators have to be multiplied by \( i \), and the commutation relations get an additional factor \( i \) on the right hand side. This convention is particularly common in the context of quantum physics.

1.2.4. Isometries

**Distances**

If the vector space is equipped with a scalar product, it induces a metrical distance between two vectors (see Eq. (1.34) on page 10)

\[
d(u, v) = ||u - v|| = \sqrt{g(u - v, u - v)} = \sqrt{g_{ij}(u^i - v^i)(u^j - v^j)}. \tag{1.75}
\]

In a curvilinear coordinate system, if we insert Eq. (1.51) into this definition the position of the origin \( a \) drops out. Therefore, in the coordinate basis, the length of a line between two points with the coordinates \( x^1, \ldots, x^d \) and \( y^1, \ldots, y^d \) is given by

\[
\ell(x, y) = \sqrt{g_{ij}(x^i - y^i)(x^j - y^j)}. \tag{1.76}
\]

In a positive definite metric the radicand is always non-negative, ensuring that \( \ell \geq 0 \). In Special Relativity, however, where we will be dealing with a non-positive-definite pseudo metric, the radicand can be negative, resulting in an imaginary distance. For this reason it is more convenient to consider the **squared distance** rather than the distance itself:

\[
\ell^2(x, y) = g_{ij}(x^i - y^i)(x^j - y^j). \tag{1.77}
\]

**Linear isometry**

Let us consider a linear coordinate transformation of the form

\[
\tilde{x}^i = \Lambda^i_j x^j \tag{1.78}
\]

A linear isometric coordinate transformation or short **linear isometry** is a linear coordinate transformation that preserves the scalar product and therewith distances and angles. As the scalar product induces norm and distance, this means that isometries are also distance-preserving maps.

In practice, an isometry is a transformation that leaves the metric tensor invariant:

\[
g_{kl} = \tilde{g}_{kl}, \quad \tilde{g}^{kl} = \tilde{g}^{kl}. \tag{1.79}
\]

Isometries preserve the metric tensor, i.e., angles and distances. Simple examples: Translations, reflections and rotations in \( \mathbb{R}^n \).
Proof: For proving this relation it makes a difference whether the coordinate transformation is an active or a passive transformation:

- In the case of a passive coordinate transformation, the scalar product of vectors with the same coordinates should be the same in the basis \( \{ e_i \} \) and the basis \( \{ \tilde{e}_i \} \):
  \[
g(x^i e_i, y^j e_j) = g(x^i \tilde{e}_i, y^j \tilde{e}_j) \Rightarrow g_{ij} x^i y^j = \tilde{g}_{ij} x^i y^j \Rightarrow g_{ij} = \tilde{g}_{ij}
  \]

- In the case of an active coordinate transformation \( x \mapsto \tilde{x} : x^i \mapsto \tilde{x}^i = \Lambda^i_j x^j \) the scalar product should unchanged, i.e. \( g(x, y) = g(\tilde{x}, \tilde{y}) \):
  \[
g_{ij} x^i y^j = \tilde{g}_{ij} \underbrace{\Lambda^i_k x^k}_{=g_{ij}} \Rightarrow g_{ij} = \tilde{g}_{ij}.
  \]

The second identity \( g^{kl} = \tilde{g}^{kl} \) follows from the first one by matrix inversion. \( \Box \)

This completes our mathematical survey. Now we are ready to connect what we have learned so far with the physical reality in the following chapter.
2. Minkowski Spacetime

In this Chapter we examine the structure of space and time in the framework of the Theory of Special Relativity. This applies to all situations where gravity can be neglected. As we will see below, space and time form a single entity, called spacetime, a 3+1-dimensional vector space with a particular metric, the so-called Minkowski metric. The aim of this Chapter is to get familiar with the Minkowski space and to understand its properties.

2.1. Lorentz transformation

2.1.1. From Galilei to Lorentz invariance

Galilei invariance
As we will see in the following, the laws of Newtonian mechanics are invariant under Galilei transformations. A Galilei transformation is a coordinate transformation from the coordinates $x, t$ of an observer in the rest frame to the coordinates $\tilde{x}, \tilde{t}$ in the frame of another observer who is with moving with constant velocity $v$:

$$
\begin{align*}
    x &\rightarrow \tilde{x} = x - vt \\
    t &\rightarrow \tilde{t} = t
\end{align*}
$$

In matrix form, this transformation can also be written as

$$
\begin{pmatrix}
    \tilde{t} \\
    \tilde{x} \\
    \tilde{y} \\
    \tilde{z}
\end{pmatrix} =
\begin{pmatrix}
    1 & 0 & 0 & 0 \\
    -v_x & 1 & 0 & 0 \\
    -v_y & 0 & 1 & 0 \\
    -v_z & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
    t \\
    x \\
    y \\
    z
\end{pmatrix}
$$

or in short

$$
\begin{pmatrix}
    \tilde{t} \\
    \tilde{x}
\end{pmatrix} =
\begin{pmatrix}
    1 & 0 \\
    -v & 1
\end{pmatrix}
\begin{pmatrix}
    t \\
    x
\end{pmatrix}
$$

The Galilei transformation has the following key properties:

- It is a linear transformation.
- The time variable $t$ is not modified, it is identical for all observers.

Therefore, in the Newtonian world, time is a universal quantity which takes the same value everywhere in the universe. It is therefore a global parameter rather than a coordinate, parameterizing the evolution of the entire Universe.
Electrodynamical wave equation

In 1864 the Scottish mathematical physicist J.C. Maxwell formulated a unified theory of magnetism and electricity in the form of four partial differential equations. His most notable achievement was to predict the existence of electromagnetic waves. Let us briefly summarize the main facts:

The four Maxwell equations in vacuum read:

\[
\begin{align*}
\nabla \cdot E &= 0 \\
\nabla \times E &= -\frac{\partial B}{\partial t} \\
\nabla \cdot B &= 0 \\
\nabla \times B &= \mu_0 \varepsilon_0 \frac{\partial E}{\partial t}
\end{align*}
\]

Taking the curl of the second and fourth equation yields:

\[
\begin{align*}
\nabla \times (\nabla \times E) &= -\frac{\partial}{\partial t} \nabla \times B = -\mu_0 \varepsilon_0 \frac{\partial^2 E}{\partial t^2} \\
\nabla \times (\nabla \times B) &= \mu_0 \varepsilon_0 \frac{\partial}{\partial t} \nabla \times E = -\mu_0 \varepsilon_0 \frac{\partial^2 B}{\partial t^2}
\end{align*}
\] (2.4)

On the other hand, we can use the vector identities

\[
\begin{align*}
\nabla \times (\nabla \times B) &= \nabla \left( \nabla \cdot B \right) - \nabla^2 B, \\
\nabla \times (\nabla \times E) &= \nabla \left( \nabla \cdot E \right) - \nabla^2 E
\end{align*}
\] (2.5)

which leads to the so-called wave equation

\[
\begin{align*}
\mu_0 \varepsilon_0 \frac{\partial^2 B}{\partial t^2} &= \nabla^2 B, \\
\mu_0 \varepsilon_0 \frac{\partial^2 E}{\partial t^2} &= \nabla^2 E
\end{align*}
\] (2.6)

where

\[
c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = 2.99792458 \times 10^8 \text{ m/s}
\] (2.7)

is the speed of light in vacuum, a new fundamental constant that enters Maxwell’s theory. Defining an abbreviation named quabla by\footnote{In many textbooks, $\Box$ is defined with a different sign due to an ambiguity in the definition of the metric, as will be explained below.}

\[
\Box := \nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2}
\] (2.8)

the wave equation can be written in the ultra-compact form

\[
\Box B = 0, \quad \Box E = 0.
\] (2.9)

Note that each component of the magnetic and the electric field satisfies the wave equation separately.
2.1 Lorentz transformation

**In the wave equation Galilei-invariance is violated**

In Sect. 1.2.2 on page 19 we learned that under linear coordinate transformations, the corresponding derivatives transform *inversely*. Therefore, with the Galilei transformation given in Eq. (2.3), the corresponding derivatives will transform as

\[
\left( \partial_t, \nabla \right) = \left( \partial_t, \nabla \right) \begin{pmatrix} 1 & 0 \\ +v & 1 \end{pmatrix}
\]

(2.10)

or explicitly

\[
\tilde{\partial}_t = \partial_t + v \cdot \nabla, \quad \tilde{\nabla} = \nabla
\]

(2.11)

**Proof:** You can prove (2.11) by checking that the algebra for coordinates and partial derivatives in (1.56) is fulfilled in both the old coordinates \(t, x\) and the new coordinates \(\tilde{t}, \tilde{x}\).

**Remark:** If we compare this transformation law with the original form of the Galilei transformation, where we pointed out that time is not transformed at all, it may be at first glance somewhat surprising that the time derivative changes while the spatial gradient \(\nabla\) remains unchanged. However, imagine that you are sitting in a train. The slope of a hill (the gradient) is always the same, irrespective of whether the train is moving or not. However, the height of the landscape that you can see begins to change as soon as the train moves.

Now it is easy to calculate how the operator \(\Box\) changes under Galilei transformation. Here we find:

\[
\Box = \Box - \frac{1}{c^2} \left( 2\partial_t (v \cdot \nabla) - (v \cdot \nabla)^2 \right)
\]

(2.12)

In the 1870’s, this came not as a surprise because the wave equation \(\Box f = 0\) was already known, for example in the context of sound waves. All the known wave phenomena had in common that they took place on some kind of carrier, some kind of vibrating medium. The existence of such a medium would of course single out a particular reference frame, the so-called *rest frame* in which the medium is locally at rest. The wave equation is expected to be correct only in the rest frame of the medium, while in other frames additional terms have to be taken into account. Initially, most physicists thought that the story with electromagnetic waves would be just the same.

But where is the medium responsible for electromagnetic waves? If there was such a medium immersed in the Universe, we should be able to detect it as we are moving around the sun. This led to the famous experiment by Michelson and Morley in 1887, who built in interferometer designed to detect the “ether wind” caused by earth’s motion. But surprisingly, the result was negative. Apparently there is no medium in which the electromagnetic waves propagate. Rather the experiment indicated that the velocity of light is the same – in any reference frame. This implied that the Maxwell equations and the wave equation are correct in all frames. But clearly, this was in con-
tradiction with the traditional concept of Galilei invariance because of the emerging extra term in Eq. (2.12).

2.1.2. Derivation of the Lorentz transformation in 1+1 dimensions

Invariance of the wave equation

Accepting that the electromagnetic wave equation is not invariant under Galilei transformations, we could ask the question if we could replace it by another linear transformation under which the wave equation does not change. This was the question addressed by the Dutch mathematical physicist Hendrik Antoon Lorentz.

For simplicity let us consider the problem in a 1+1-dimensional spacetime, where the physical position space with coordinate $x$ is only one-dimensional. The idea is to replace the Galilei transformation (2.3), namely,

\[
\begin{pmatrix}
\tilde{t} \\
\tilde{x}
\end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -v & 1 \end{pmatrix} \begin{pmatrix} t \\
x \end{pmatrix}
\]  

by a more general linear transformation of the form

\[
\begin{pmatrix}
\tilde{t} \\
\tilde{x}
\end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} t \\
x \end{pmatrix}
\]  

With this assumption we again find that the derivatives transform inversely, i.e.

\[
\begin{pmatrix}
\partial_t \\
\partial_x
\end{pmatrix} = \frac{1}{AD - BC} \begin{pmatrix} D & -B \\ -C & A \end{pmatrix} \begin{pmatrix}
\partial_t \\
\partial_x
\end{pmatrix}
\]  

or explicitly

\[
\begin{align*}
\partial_t &= D \partial_t - C \partial_x \\
\partial_x &= A \partial_x - B \partial_t
\end{align*}
\]  

Hence

\[
\Box = \tilde{\partial}_x^2 - \frac{1}{c^2} \tilde{\partial}_t^2 = \frac{\left( A^2 - \frac{c^2}{c^2} \right) \partial_x^2 - 2 \left( AB - \frac{CD}{c^2} \right) \partial_x \partial_t + \left( B^2 - \frac{D^2}{c^2} \right) \partial_t^2}{(AD - BC)^2}
\]  

If we require invariance $\Box = \Box$ we get three equations

\[
\frac{A^2 - \frac{c^2}{c^2}}{(AD - BC)^2} = 1, \quad AB - \frac{CD}{c^2} = 0, \quad \frac{B^2 - \frac{D^2}{c^2}}{(AD - BC)^2} = -\frac{1}{c^2}
\]  

with the simple solution

\[
A = D, \quad C = Bc^2, \quad AD - BC = 1.
\]
2.1 Lorentz transformation

**Physical interpretation**

Consider the origin of the new coordinate system $\tilde{x} = 0$. How does this origin move as a function of time in the original coordinate system? To answer this question we first invert the transformation Eq. (2.14):

\begin{align*}
\begin{pmatrix}
\tilde{t} \\
\tilde{x}
\end{pmatrix} = \frac{1}{AD - BC} \begin{pmatrix} D & -B \\ -C & A \end{pmatrix} \begin{pmatrix} t \\
\tilde{x}
\end{pmatrix}
\end{align*}

(2.20)

Inserting $\tilde{x} = 0$ on the r.h.s. we get

\begin{align*}
t = \frac{D\tilde{t}}{AD - BC}, \quad \tilde{x} = -\frac{C\tilde{t}}{AD - BC}.
\end{align*}

(2.21)

From this we read off that the origin of the new coordinate system moves at the constant velocity

\begin{align*}
v = \frac{x}{t} = -\frac{C}{D}.
\end{align*}

(2.22)

The transformation therefore describes a change of the reference frame between an observer at rest and an observer moving at constant velocity $v$, or in other words, between two *inertial systems* moving relative to each other with velocity $v$.

**Lorentz transformation derived from the invariance of the wave equation**

Combining Eq. (2.22) with the three equations in (2.19) we have altogether a system of four equations with four unknowns $A, B, C, D$. The solution reads

\begin{align*}
A = D = \frac{1}{\sqrt{1 - \beta^2/c^2}}, \quad B = -\frac{v}{c^2\sqrt{1 - \beta^2/c^2}}, \quad C = -\frac{v}{\sqrt{1 - \beta^2/c^2}}.
\end{align*}

(2.23)

With the definition of the dimensionless *Lorentz factor*

\begin{align*}
\gamma := \frac{1}{\sqrt{1 - \beta^2}}, \quad \beta := \frac{v}{c}
\end{align*}

(2.24)

the Lorentz transformation reads

\begin{align*}
\begin{pmatrix} \tilde{t} \\
\tilde{x}
\end{pmatrix} = \gamma \begin{pmatrix} 1 & -\frac{v}{c^2} \\ -\frac{v}{c} & 1 \end{pmatrix} \begin{pmatrix} t \\
x
\end{pmatrix}
\end{align*}

(2.25)

The Lorentz factor $\gamma$ measures how severe the relativistic corrections are. As can be seen in the adjacent figure, for $v/c < 0.1$ we have $\gamma \approx 1$, meaning that for velocities below 10% of the velocity of light, relativistic corrections are expected to be negligible. However, for more than 90% of $c$, we can expect the corrections to be severe.

In this context it is instructive to check the non-relativistic limit $c \to \infty$, where $\gamma \to 1$: 
Here the transformation reduces to the ordinary Galilei transformation \[ (2.3) \]

\[
\lim_{c \to \infty} \begin{pmatrix} t \\ \tilde{x} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -v & 1 \end{pmatrix} \begin{pmatrix} t \\ x \end{pmatrix}.
\] (2.26)

This is very nice since it already indicates that we recover Galileian invariance in the limit \( c \to \infty \) (or \( v \to 0 \), which is just the same).

Unlike the Galilei transformation, the Lorentz transformation fully mixes space and time. For this reason it is meaningful to interpret \( t \) and \( x \) as coordinates in a common vector space, called spacetime. However, the units of \( t \) and \( x \) are different, and it does not make much sense to define vector components with different units. But this problem can be circumvented easily by writing \( ct \) instead of \( t \), and likewise \( c\tilde{t} \) instead of \( \tilde{t} \). This leads to the usual form of the Lorentz transformation in 1+1 dimensions:

\[
\begin{pmatrix} c\tilde{t} \\ \tilde{x} \end{pmatrix} = \begin{pmatrix} \gamma & -\beta \gamma \\ -\beta \gamma & \gamma \end{pmatrix} \begin{pmatrix} ct \\ x \end{pmatrix}.
\] (2.27)

This formula can be inverted easily, giving

\[
\begin{pmatrix} ct \\ x \end{pmatrix} = \begin{pmatrix} \gamma & \beta \gamma \\ +\beta \gamma & \gamma \end{pmatrix} \begin{pmatrix} c\tilde{t} \\ \tilde{x} \end{pmatrix}.
\] (2.28)

The professional ethic of theoretical physicists stipulates that the speed of light should be set to \( c = 1 \). However, in the following we will continue to keep \( c \) in the formulas.

**Events**

In Newtonian mechanics, time is a global parameter and the points in space are understood as positions. Contrarily, in the Minkowski space, time is a coordinate on equal footing with the spatial coordinates. A point in spacetime therefore specified a position combined with a certain instance of time. For this reason, points in spacetime are referred to as *events*.

**Conventional derivation of the Lorentz transformation**

In most textbooks the Lorentz transformation is derived from the assumption that light propagates at the same speed \( c \) in any inertial reference frame. Let us briefly summarize this standard derivation although it is a bit cumbersome. The first part of the derivation is shown in Fig. 2.1:

- The left panel shows the spacetime coordinate system in the laboratory frame \( S \) with two particular *events*, namely, \( A \) at the origin and \( B \) at the coordinates \((ct, x)\). We assume that both events are connected by a light ray, meaning that \( x = ct \).

- We assume that two other observers are moving to the right along the dashed green lines at constant velocity \( v < c \), one of them passing \( A \) and the other one passing \( B \). Clearly, the spatial distance of the two observers in the laboratory frame is \( a = ct - vt \).
2.1 Lorentz transformation

The proof relies on the assumption that $\tilde{a}$ is larger than $a$. More specifically, it is assumed that all length scales are contracted by factor $\delta > 1$ that depends only on the relative velocity $v/c$. This means that

$$\tilde{a} = \delta a.$$  \hfill (2.29)

Knowing that $a = (c - v)t$ and $\tilde{a} = c\tilde{t}$ this yields the relation

$$\tilde{t} = \delta \left(1 - \frac{v}{c}\right)t.$$  \hfill (2.30)

Now it is important to recognize that frame changes are reflexive: We can use the same chain of arguments in opposite direction, the only difference being that $v$ has to be replaced by $-v$. This brings us directly to

$$t = \delta \left(1 + \frac{v}{c}\right)\tilde{t}.$$  \hfill (2.31)

Inserting this equation into the previous one yields the identity $t = \delta^2 (1 - v^2/c^2)t$, meaning that the length contraction factor is given by $\delta = \pm \frac{1}{\sqrt{1 - v^2/c^2}}$. Since the minus sign would reflect the time coordinate, we restrict ourselves to the positive solution. Hence the expression on the right side just coincides with the Lorentz factor $\gamma$ introduced in Eq. (2.24):

$$\delta = \gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}.$$  \hfill (2.32)

Now we proceed by considering a general event at $(ct, x)$ that is not connected to the origin by a light ray. Again we introduce two observers moving to the right at velocity $v$, one
passing the origin and the other one passing the event at \((ct, x)\). As can be seen in the figure, their spatial distance in the laboratory frame \(S\) is

\[ a = x - vt. \]

However, in the co-moving frame, where the two observers are at rest, their spatial distance \(\tilde{a} = \gamma a\) is larger. Since \(\tilde{x} = \tilde{a}\) we get the transformation law

\[ \tilde{x} = \gamma(x - vt) \] (2.33)

Since the transformation has to be reflexive (see above), we can swap \(x \leftrightarrow \tilde{x}, t \leftrightarrow \tilde{t}\) and \(v \leftrightarrow -v\), obtaining

\[ x = \gamma(\tilde{x} - v\tilde{t}). \] (2.34)

Inserting (2.34) into (2.33) and solving for \(\tilde{t}\) one obtains

\[ \tilde{t} = \gamma(t - \frac{v}{c}x) \] (2.35)

and by reflexivity

\[ t = \gamma(\tilde{t} + \frac{v}{c}\tilde{x}). \] (2.36)

These results are in agreement with Eq. (2.27), namely

\[
\begin{align*}
\tilde{t} &= \gamma(ct - \beta x) \\
\tilde{x} &= \gamma(x - \beta ct)
\end{align*}
\]

where \(\beta = v/c\) and \(\gamma = 1/\sqrt{1 - \beta^2}\). This completes the traditional proof of the Lorentz transformation.

### 2.1.3. Properties of the Lorentz transformation in 1+1 dimensions

**Infinitesimal Lorentz transformations**

The set of all Lorentz transformations in 1+1 dimensions forms a group, where the group elements are controlled by a single continuous parameter, namely, the relative velocity of the inertial reference frames \(\beta = v/c\). We can easily Taylor-expand these group elements around the identity \((\beta = 0)\), and this tells us that the group of Lorentz transformations is a Lie group (cf. Sect. 1.2.3 on page 21). In order to understand its structure, let us now consider infinitesimal Lorentz transformations.

In the limit \(v \ll c\), where \(\beta = v/c \ll 1\), we can expand the Lorentz factor by

\[ \gamma = 1 + \frac{1}{2}\beta^2 + \frac{3}{8}\beta^4 + \mathcal{O}(\beta^6). \] (2.38)

Since the lowest nontrivial contribution is already of second order, the first-order approximation of the Lorentz transformation (2.28) can be obtained by setting \(\gamma \approx 1\),
2.1 Lorentz transformation

Giving

\[
\begin{pmatrix}
ct \\
x
\end{pmatrix} = \begin{pmatrix}
1 + \beta & +\beta \\
+\beta & 1
\end{pmatrix}
\begin{pmatrix}
ct' \\
x'
\end{pmatrix} = \begin{pmatrix}
1 + \beta & 0 \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
ct' \\
x'
\end{pmatrix} + \mathcal{O}(\beta^2)
\] (2.39)

or in short

\[x = (1 + \beta \lambda) \hat{x}.\] (2.40)

Obviously, \(\lambda\) is the generator of the Lorentz transformation in 1+1 dimensions, obeying the simple algebra \(\lambda^2 = 1\). This means that the full group of Lorentz transformations is a Lie group with group elements given by a matrix exponential function:

\[x = \exp(\theta \lambda) \hat{x}\] (2.41)

Here \(\theta\) is a parameter that is in some sense analogous to the angle \(\phi\) in a rotation (see Eq. (1.70) on page 22). A simple calculation (see exercises) allows us to compute the exponential function explicitly, the result being

\[
\begin{pmatrix}
ct \\
x
\end{pmatrix} = \begin{pmatrix}
\cosh \theta & \sinh \theta \\
\sinh \theta & \cosh \theta
\end{pmatrix}
\begin{pmatrix}
ct' \\
x'
\end{pmatrix} = \Lambda(\theta)
\] (2.42)

Recall: The definitions of the trigonometric and the hyperbolic functions are given by

\[
\cos \alpha = \frac{e^{i\alpha} + e^{-i\alpha}}{2}, \quad \sin \alpha = \frac{e^{i\alpha} - e^{-i\alpha}}{2i}, \quad \cosh \theta = \frac{e^\theta + e^{-\theta}}{2}, \quad \sinh \theta = \frac{e^\theta - e^{-\theta}}{2},
\]

In Special Relativity, the parameter \(\theta\) is known as the rapidity. Note that unlike rotational angles, it is not restricted to \([0, 2\pi]\). To find out how the rapidity is related to the relative velocity of the two inertial systems \(\beta = v/c\), we only have to equate Eq. (2.42) and (2.28), giving the relations

\[
cosh \theta = \gamma, \quad \sinh \theta = \gamma \beta.
\] (2.43)

Dividing the second equation by the first one yields \(\tanh \theta = \beta\), hence

\[
\theta = \tanh^{-1}\left(\frac{v}{c}\right).
\] (2.44)

Remember: A Lorentz transformation in 1+1 dimensions can be written in the form

\[
\begin{pmatrix}
ct \\
x
\end{pmatrix} = \begin{pmatrix}
\cosh \theta & \sinh \theta \\
\sinh \theta & \cosh \theta
\end{pmatrix}
\begin{pmatrix}
ct' \\
x'
\end{pmatrix} \quad \text{with} \quad \theta = \tanh^{-1}\left(\frac{v}{c}\right).
\]

Addition theorem of velocities

Using the addition theorem of the hyperbolic functions

\[
\begin{align*}
sinh(\theta_1 \pm \theta_2) &= \sinh(\theta_1) \cdot \cosh(\theta_2) \pm \sinh(\theta_2) \cdot \cosh(\theta_1) \\
cosh(\theta_1 \pm \theta_2) &= \cosh(\theta_1) \cdot \cosh(\theta_2) \pm \sinh(\theta_1) \cdot \sinh(\theta_2)
\end{align*}
\] (2.45)
it is easy to check that
\[
\begin{pmatrix}
  \cosh \theta_2 & \sinh \theta_2 \\
  \sinh \theta_2 & \cosh \theta_2
\end{pmatrix}
\begin{pmatrix}
  \cosh \theta_1 & \sinh \theta_1 \\
  \sinh \theta_1 & \cosh \theta_1
\end{pmatrix}
= \begin{pmatrix}
  \cosh(\theta_1 + \theta_2) & \sinh(\theta_1 + \theta_2) \\
  \sinh(\theta_1 + \theta_2) & \cosh(\theta_1 + \theta_2)
\end{pmatrix}
\]
(2.46)
or in short
\[
\Lambda(\theta_2)\Lambda(\theta_1) = \Lambda(\theta_1 + \theta_2),
\]
(2.47)
meaning that the rapidity parameter \( \theta \) is additive under concatenation of Lorentz transformations in 1+1 dimensions. Therefore, if we perform a sequence of such Lorentz transformations labeled from 1 to \( N \), this amounts to the same as a single Lorentz transformation with
\[
\theta = \sum_{i=1}^{N} \theta_i \implies \frac{v}{c} = \tanh \sum_{i=1}^{N} \tanh^{-1} \left( \frac{v_i}{c} \right)
\]
(2.48)
In particular, for the concatenation of two subsequent transformations \( (N = 2) \) we get
\[
\frac{v}{c} = \tanh \left[ \tanh^{-1} \left( \frac{v_1}{c} \right) + \tanh^{-1} \left( \frac{v_2}{c} \right) \right]
\]
(2.49)
Using \( \tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}} \) and \( \tanh^{-1} y = \frac{1}{2} \ln \left( \frac{1+y}{1-y} \right) \) this reduces to the famous relativistic addition theorem for velocities:
\[
\beta = \frac{\beta_1 + \beta_2}{1 + \beta_1 \beta_2} \implies v = \frac{v_1 + v_2}{1 + \frac{v_1 v_2}{c^2}}
\]
(2.50)

2.1.4. Lorentz transformation in 3+1 dimensions

**Lorentz boosts**

So far we restricted ourselves to the (nonphysical) case of 1+1 dimensions. The real position space of our Universe, however, is known to be three-dimensional. In this case the relative motion of the two inertial system is characterized by a velocity vector \( \vec{v} \in \mathbb{R}^3 \). This vector specifies the direction of the so-called Lorentz boost.

If the boost takes place in the \( x \)-direction, i.e. \( \vec{v} = ve_x \), and if we consider events with the coordinates \( y = z = 0 \), all results derived above remain valid. But what happens if the events differ in their \( y \) or \( z \) coordinates? Will distances in \( y \) or \( z \) direction also contract under Lorentz boosts in the \( x \)-direction? This is the question we would like to address in the following.

**Physicists derivation**

To answer this question, let us consider a train moving with constant velocity \( v \). Assume that a light ray oscillates vertically in the \( y \)-direction between mirrors at the floor and the ceiling, as shown in Fig. 2.2. The right panel shows the situation in the co-moving frame of the train: Here the light ray simply goes up and down. If the mirrors are separated by the distance \( \tilde{b} \) in the frame of the train, the time \( 2\tilde{t} \) for going forth and back
(measured on the train) is simply
\[ 2\bar{\tau} = 2\tilde{b}/c. \quad (2.51) \]

Now let us consider the same situation in the rest frame of the train station. From this perspective the mirrors move to the right at constant velocity \( v \) and the light rays perform some kind of zigzag. Because of the time dilation law, the time \( \tau \) to go forth and back will be longer than \( \tilde{\tau} \):
\[ \tau = \gamma \tilde{\tau}, \quad \gamma = \frac{1}{\sqrt{1-v^2/c^2}}. \quad (2.52) \]

On the other hand, the length of the light ray between two reflections is \( c\tau \), hence
\[ b^2 = c^2\tau^2 - v^2\tau^2 = c^2 \frac{\tau^2}{\gamma^2} = c^2 \tau^2 = \tilde{b}^2. \quad (2.53) \]

From this we read off that \( b = \tilde{b} \), so we can conclude that the dimensions perpendicular to the boost direction are not affected by the Lorentz transformation.

**A Lorentz boost does not affect the degrees of freedom perpendicular to the boost direction.**

This allows us to immediately write down the Lorentz boost in \( x \)-direction in 3+1 dimension which generalizes the formula given in Eq. (2.42):
\[
\begin{pmatrix}
  ct \\
  x \\
  y \\
  z
\end{pmatrix}
= \begin{pmatrix}
  \cosh \theta & \sinh \theta \\
  \sinh \theta & \cosh \theta \\
  1 & 0 \\
  0 & 1
\end{pmatrix}
\begin{pmatrix}
  \tilde{c}t \\
  \tilde{x} \\
  \tilde{y} \\
  \tilde{z}
\end{pmatrix}
= \Lambda_x(\theta) \quad (2.54)
\]

Similar expressions can be derived for Lorentz boosts in \( y \) and in \( z \)-direction.

**Brute-force derivation***

The behavior in higher dimensions can also be ‘derived’ by brute force with an algebraic computer system like *Mathematica®* as demonstrated on the right side. This proof
requires knowledge of the isometry condition that will be discussed below in Eq. (2.81) on page 42. If you are not yet familiar with the isometry condition, you can skip this paragraph.

Let us consider a Lorentz transformation, for simplicity here only in in 2+1 dimensions, with the known behavior in \( ct, x \) and with unspecified matrix elements in the third row and column:

\[
\Lambda = \begin{pmatrix}
cosh \theta & \sinh \theta & \Lambda_{13} \\
\sinh \theta & \cosh \theta & \Lambda_{23} \\
\Lambda_{31} & \Lambda_{32} & \Lambda_{33}
\end{pmatrix}
\]

Then we simply solve the isometry equation

\[
\Lambda^\mu \eta^{\nu \tau} \Lambda^{\tau \rho} = \eta^{\mu \nu}
\]

which can be expressed in matrix form as \( \Lambda \eta \Lambda^T = \eta \). As can be seen, the computer finds two solutions, one with the \( y \)-component unchanged and one where it is reflected, but in both cases we do not have a contraction or any non-trivial mixing of the components in \( y \)-direction. This confirms our findings above. The same can be done in 3+1 dimensions.

### 2.2. Minkowski space

#### 2.2.1. Space and time unified in a single vector space

**4-vectors**

Lorentz transformations fully mix space and time. Therefore, it is meaningful to unify space and time in a common 1+3 = 4-dimensional vector space. In the standard basis, the elements of this vector space, called **4-vectors**, have the components

\[
\begin{pmatrix}
ct \\
x \\
y \\
z
\end{pmatrix} = \begin{pmatrix}
x^0 \\
x^1 \\
x^2 \\
x^3
\end{pmatrix}
\]  
(2.55)

While Latin indices \( i, j, k, \ldots \) are usually used for spatial components running from 1 to 3, most textbooks take the convention of using Greek letters \( \mu, \nu, \ldots \) for indices running from 0 to 3, including the time coordinate.

While an ordinary 3-vector marks a particular position in space, a 4-vector marks a particular point in space at a particular time. Such a point is spacetime is denoted as an **event**. It is important to note that an event is some kind of physical reality, modeled by
2.2 Minkowski space

a vector \( x \in \mathbb{R}^{1+3} \) in the 1+3-dimensional spacetime. This vector can be represented in different ways depending on the chosen basis \( e_\mu \) by

\[
x = x^\mu e_\mu
\]  
(2.56)

In the Cartesian standard basis, the representation reads

\[
x = x^\mu e_\mu = c t e_0 + x e_1 + y e_2 + z e_3.
\]  
(2.57)

**Linear forms**

As in the previous section, for every given basis \( \{ e_\mu \} \) there is a corresponding dual basis \( \{ e^\nu \} \) in the covector space. Linear forms in the covector space can be represented by components with lower indices as \( \alpha = e^\nu \alpha_\nu \). The most important example is the gradient operator

\[
\nabla = e^\nu \partial_\nu .
\]  
(2.58)

In the standard basis the gradient is given by

\[
\left( \partial_0, \partial_1, \partial_2, \partial_3 \right) = \left( \frac{1}{c} \frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right)
\]  
(2.59)

**Scalar product: The Minkowski metric**

The **Minkowski space** is a space of 4-vectors (together with the corresponding co-vector space) which is equipped with a special metric \( g_{\mu\nu} \), the so-called **Minkowski metric**. This metric is so special that we shall denote it by a different letter, namely, by \( \eta_{\mu\nu} \) instead of \( g_{\mu\nu} \):

\[
\eta_{\mu\nu} = \begin{pmatrix}
-1 & +1 & +1 & +1 \\
+1 & 1 & +1 & +1 \\
+1 & +1 & 1 & +1 \\
+1 & +1 & +1 & 1
\end{pmatrix}
\]  
(2.60)

The reason is that in General Relativity, the gravitational field is encoded in the non-diagonal elements of the metric tensor. Only without gravity the metric is diagonal and has this particular form. By using the notation \( \eta_{\mu\nu} \) we remind ourselves that we are dealing with zero gravity in the framework of Special Relativity.

**Remark:** In the literature one finds two different conventions: The “mostly minus” convention \( \eta_{\mu\nu} = \text{diag}(1, -1, -1, -1) \) and “mostly plus” convention \( \eta_{\mu\nu} = \text{diag}(-1, +1, +1, +1) \). Both choices are formally equivalent if we adapt various signs in the formulas. Here we use “mostly plus” because it is also widely used in advanced topics such as string theory and AdS/CFT correspondence.

Note that the Minkowski metric is self-inverse:

\[
\eta_{\mu\nu} = \eta^{\mu\nu}.
\]  
(2.61)

That is, in Special Relativity one can raise and lower indices simply by switching the
Spacelike, timelike, and lightlike distances

The Minkowski metric carries the signature $- + + +$, hence it is a pseudometric where the scalar product is no longer positive definite and where nonzero vectors may have zero length. The set of all position vectors with vanishing norm $|x| = 0$ constitutes the so-called light cone, as shown in Fig. 2.3.

The light cone separates the Minkowski space into three parts, namely, a timelike future, a timelike past, and a spacelike region in outside the light cone. More specifically, let $A$ and $B$ be two events represented by the vectors $a$ and $b$ and let $x = a - b$ be the difference vector. According to Eq. (1.77) on page 23 the squared distance between the two events is given by

$$\ell^2(a, b) = \eta_{\mu\nu} (a^\mu - b^\mu) (a^\nu - b^\nu) = \eta_{\mu\nu} x^\mu x^\nu = x^\mu x^\mu. \quad (2.62)$$

In the “mostly plus” convention, we distinguish the following cases:

- $\ell^2 > 0$: Spacelike distance: The two events cannot be causally connected.
- $\ell^2 = 0$: Lightlike distance: The two events can be connected by a light ray.
- $\ell^2 < 0$: Timelike distance: The two events can be connected causally.

Two events which can be connected by a light ray have the relativistic distance zero.

Example: Solving the wave equation in Minkowski space

Compared to the usual notation with separate space and time, as for example in introductory textbooks on electrodynamics, the covariant notation with 4-vectors in the 1+3-dimensional Minkowski space turns out to be extremely beautiful, leading to compact
and very aesthetic formulas. As an example let us consider again the wave equation (see Eq. (2.9) on page 26)

\[ \Box \Phi = 0, \]  

(2.63)

where \( \Phi(t, x, y, z) \) stands for some field (for example \( E \) or \( B \)) and where ‘quabla’ is defined by

\[ \Box = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2}. \]  

(2.64)

First of all it is easy to check that the d’Alembert operator ‘quabla’ can be expressed as

\[ \Box = \partial_\mu \partial^\mu \]  

(2.65)

\textbf{Proof:} \( \partial_0 := \frac{1}{c} \frac{\partial}{\partial t} \) and \( \partial_i := \frac{\partial}{\partial x^i} \) \( \Rightarrow \) \( \Box = -\partial_0^2 + \partial_1^2 + \partial_2^2 + \partial_3^2 = \eta^{\mu\nu} \partial_\mu \partial_\nu = \partial_\mu \partial^\mu. \)

In order to find a solution, let us try a plane-wave ansatz of the form

\[ \Phi(x) = e^{i k \cdot x} = e^{ik_\mu x^\mu}. \]  

(2.66)

Inserting this ansatz into the wave equation we find

\[ \Box \Phi = \eta^{\rho\tau} \partial_\rho \partial_\tau \exp(i \eta_{\mu\nu} k^\mu x^\nu) = \eta^{\rho\tau} i^2 \eta_{\rho\mu} k^\mu \eta_{\tau\nu} k^\nu \exp(i \eta_{\mu\nu} k^\mu x^\nu) \]

\[ = \Box = \Phi(x) \]

\[ = -\eta_{\rho\mu} k^\mu \delta_\rho^\nu \exp(i \eta_{\mu\nu} k^\mu x^\nu) = -k_\rho k^\rho \Phi. \]

(2.67)

Hence the plane wave ansatz is indeed a solution of the wave equation \( \Box \Phi = 0 \) provided that the 4-vector \( k \) lies on the light cone:

\[ k_\rho k^\rho = 0. \]  

(2.68)

The 4-vector \( k \) is a four-dimensional analog of the wave number vector \( \vec{k} \), including the angular frequency in the zeroth component:

\[ k = \begin{pmatrix} k^0 \\ k_1 \\ k_2 \\ k_3 \end{pmatrix} = \begin{pmatrix} \omega/c \\ k_x \\ k_y \\ k_z \end{pmatrix} \]  

(2.69)

or in short

\[ k = \begin{pmatrix} \omega/c \\ \vec{k} \end{pmatrix} \]  

(2.70)

That is, in Special Relativity the angular frequency \( \omega \) and the wave number \( \vec{k} \) are unified in a single object, a 4-wave-number \( \mathbf{k} \). The requirement \( k_\rho k^\rho = 0 \) (telling us that \( k \) has to lie on the light cone) means that the angular frequency and the wave number 3-vector

\[ \text{in the literature the sign of } \Box \text{ depends on the convention used for the metric tensor. Really cool people, who are using the 'mostly plus' convention, define quabla by } \Box = \Delta - \frac{1}{c^2} \frac{\partial^2}{\partial t^2}. \]  

The rest of the world, including Wikipedia, uses the definition \( \Box = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \Delta. \)
are related by
\[ \frac{\omega^2}{c^2} = \vec{k} \cdot \vec{k} \Rightarrow \omega = \pm kc \] (2.71)

This is nothing but the well-known dispersion relation of light in vacuum.

**Remark:** The simple linear relationship ensures that the phase velocity \( v_p = \frac{\omega}{k} \) and the group velocity \( v_g = \frac{d\omega}{dk} \) are identical and equal to \( c \). This is the reason why a wave packet of light in vacuum (unlike quantum-mechanical wave packets) does not spread or dissolve: A flash remains a flash, even after millions of years traveling in the Universe.

**Adding mass: The Klein-Gordon equation**

In quantum mechanics, the *Klein-Gordon equation* is a relativistic wave equation analogous to the non-relativistic Schrödinger equation which describes massive spinless quantum particles. Without going into the specific physics here, let us have a quick look at the equation itself, which takes the form

\[ \Box - M^2 \psi = 0, \] (2.72)

where \( M := \frac{mc}{\hbar} \) is a mass parameter. Clearly, for \( M = 0 \) the Klein-Gordon equation reduces to the wave equation studied above. Therefore, the Klein-Gordon equation may be understood as some kind of “wave equation with mass”.

A plane-wave ansatz analogous to the one carried out above leads to the condition

\[ k_\mu k^\mu = -M^2, \] (2.73)

meaning that the 4-wave-vector \( \vec{k} \) has a negative squared norm. It is therefore a timelike vector, located on one of two hyperboloids, the so-called **mass shell**, as sketched in the adjacent figure. The corresponding dispersion relation reads

\[ \frac{\omega}{c} = \pm \sqrt{M^2 + k^2}. \] (2.74)

Here we can already see something very general: A massless object is condemned to live on the light cone; it can only propagate at the speed of light. Contrarily, massive objects can propagate with a timelike \( k \)-vector; they can rest and propagate with \( v < c \), but it is impossible for them to propagate reach \( v = c \).

---

3 People using the mostly minus convention would use a different sign: \( (\Box + M^2)\psi = 0 \).
2.2 Minkowski space

2.2.2. Isometries of the Minkowski space

*Inertial systems*

As discussed previously, an *isometry* is an active or passive coordinate transformation $x \rightarrow \tilde{x}$ that leaves the scalar product invariant. This means that the transformation does not change the components of the metric tensor:

$$g_{\mu\nu} = \tilde{g}_{\mu\nu}, \quad \tilde{g}^{\mu\nu} = \tilde{g}_{\mu\nu}. \quad (2.75)$$

For arbitrary non-linear coordinate transformations, the metric tensor transforms according to

$$\tilde{g}_{\mu\nu}(x) = g_{\rho\tau}(x) \Lambda^\rho_\mu(x) \Lambda^\tau_\nu(x), \quad \tilde{g}^{\mu\nu}(x) = \Lambda^\mu_\rho(x) \Lambda^\nu_\tau(x) g^{\rho\tau}(x) \quad (2.76)$$

where $\Lambda(x)$ is the Jacobi matrix

$$\Lambda^\rho_\mu(x) = \frac{\partial \tilde{x}^\rho}{\partial x^\mu} = \frac{\partial x^\mu}{\partial \tilde{x}^\rho} \quad (2.77)$$

which (in the case of a non-linear transformation) depends on the point $x$ at which the derivative is taken. The resulting metric tensor $g_{\mu\nu}(x)$ is then in general non-diagonal and position-dependent, and its non-diagonal elements turn out to encode accelerations or equivalently the gravitational field, as will be studied in General Relativity. In Special Relativity, however, we are interested in physical situations which are acceleration-free (zero gravity). An acceleration-free observational frame is called an *inertial system*, where $g_{\mu\nu} = \eta_{\mu\nu}$:

$$\text{Inertial system} \iff g_{\mu\nu} = \eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1). \quad (2.78)$$

Transformations between inertial systems turn out to be linear, meaning that the Jacobi matrix $\Lambda$ is constant (independent of the position). The maximal set of such linear transformations constitutes the so-called *Poincaré group*. 

---

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The Poincaré group and its subgroups

Let us first start with a rough group-theoretical overview. The group of all transformations that preserve the Minkowski metric is the so-called Poincaré group. This group comprises the following operations:

- Ordinary rotations in space
- Lorentz boosts in arbitrary directions
- Reflections in space or time
- Translations in space and time

The Poincaré group is usually denoted as $O(3,1) \times \mathbb{R}^{3+1}$. The first factor $O(3,1)$ stands for orthogonal transformations in 3+1 dimensions denoted by $\Lambda$ while the second factor stands for translations in space and time by an arbitrary 4-vector $a \in \mathbb{R}^{3+1}$. Written out in coordinates the Poincaré group includes all linear transformations of the form

$$x^\mu \mapsto \tilde{x}^\mu = \Lambda^\mu_\nu x^\nu + a^\mu$$

which leave the metric tensor invariant:

$$\eta_{\mu\nu} = \eta_{\rho\tau} \Lambda^\rho_\mu \Lambda^\tau_\nu, \quad \eta^{\mu\nu} = \Lambda^\mu_\rho \Lambda^\nu_\tau \eta^{\rho\tau}$$

This condition, the so-called isometry condition, may be rewritten in matrix form as

$$\eta = \Lambda \eta \Lambda^T,$$

where $\Lambda^T$ is the transpose of $\Lambda$ with the components $(\Lambda^T)_\nu^\mu = \Lambda^\nu_\mu$. Clearly, the invariance condition (2.81) implies that the determinant of the transformation matrix is $\pm 1$. Transformations with a positive determinant preserve the orientation of the coordinate system (in the sense of a right-handed system but now with time included) while transformations with a negative determinant involve a reflection. Therefore, as depicted in Fig. 2.4 the Poincaré group possesses a subgroup of orientation-preserving transformations with $\det(\Lambda) = +1$, referred to as special Poincaré transformations and denoted by $SO(3,1) \times \mathbb{R}^{3+1}$ with an additional ‘S’ for ‘special’in front.

Both the full and the special Poincaré group split up into a subgroup of pure translations (not shown in the figure) and Lorentz transformations, i.e., linear transformations without translations which preserve the origin of the coordinate system. This defines the Lorentz group with reflections $O(3,1)$ and the special Lorentz group without reflections $SO(3,1)$.

Remark: It is important to note that the special Lorentz group $SO(3,1)$ includes Lorentz boosts as well as ordinary rotations in space (see below). In fact, as will be shown in an exercise, the Lorentz boosts alone do not form an independent subgroup since subsequent boosts in different directions may effectively lead to some kind of rotation.

Finally, the Lorentz group and the special Lorentz group both contain subgroups of orthochronous Lorentz transformations (german: eigentliche Lorentz-Transformationen) which preserve the direction of time. This means that they do not swap future and past in the Minkowski space (cf. Fig. 2.3). These subgroups are denoted as $O^{+}(3,1)$ and $SO^{+}(3,1)$,
Preparational exercise: The special orthogonal group $SO(3)$

In order to understand the role of Lorentz transformations as isometries of Minkowski space, let us first go back to the good old three-dimensional Euclidean space $\mathbb{R}^3$ with the metric $g_{ij} = \delta_{ij}$. Here the isometry condition (2.81) reduces to

$$1 = \Lambda 1 \Lambda^T \Rightarrow \Lambda \Lambda^T = 1.$$ (2.82)

Transformations obeying this condition are called orthogonal transformations, forming the group $O(3)$. If these transformations also preserve the orientation, meaning that $\det(\Lambda) = 1$, they are denoted as a special orthogonal transformations, forming the group $SO(3)$.

The $SO(3)$ is a Lie group, meaning that we can Taylor-expand its group elements around the identity. To first order we get

$$\Lambda = 1 + \epsilon \lambda \Rightarrow \Lambda^\mu_{\nu} = \delta^\mu_{\nu} + \epsilon \lambda^\mu_{\nu}.$$ (2.83)

Inserting this into the isometry condition (2.82) and retaining only the first-order terms we get

$$\lambda + \lambda^T = 0,$$ (2.84)

meaning that the matrix $\lambda$ has to be antisymmetric. Obviously, any antisymmetric $3 \times 3$ matrix can be written as a linear combination of the following three generators:

$$\lambda_{(12)} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_{(13)} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad \lambda_{(23)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}.$$ (2.85)

Here the generator $\lambda_{(12)}$ generates a rotation in the $x^1$-$x^2$-plane around the $x^3$-axis:

$$e^{\phi \lambda_{(12)}} = \begin{pmatrix} \cos \phi & \sin \phi \\ -\sin \phi & \cos \phi \end{pmatrix} = R_{(12)}(\varphi)$$ (2.86)

Similarly $\lambda_{(13)}$ and $\lambda_{(23)}$ generate rotations around the other two axes.

In the mathematical literature it is very common to redefine the generators with an imaginary factor $-i$ in front by

$$L_i = -\frac{i}{2} \sum_{j,k=1}^{3} \epsilon_{ijk} \lambda_{jk},$$ (2.87)

where $\epsilon_{ijk}$ denotes the fully antisymmetric Levi-Civita symbols. Explicitly they read

$$L_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad L_2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, \quad L_3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$
These three generators form a *vector operator* $\vec{L}$ which is known from quantum mechanics as the *angular momentum operator*. The components of this vector operator obey the angular momentum commutation relations

$$[L_i, L_j] = i\epsilon_{ijk} L_k.$$  \hfill (2.88)

This is the *Lie algebra* of the special orthogonal group $SO(3)$. As explained in Sect. 1.2.3 on page 22 we can use this vector operator to generate rotations around the axis $\vec{\phi} \in \mathbb{R}^3$ by the angle $\phi = |\vec{\phi}|$ by taking the exponential

$$R(\vec{\phi}) = e^{i\vec{L} \cdot \vec{\phi}}.$$  \hfill (2.89)

**Note:** If you are familiar with quantum mechanics, you may have noticed that the Pauli matrices, which are the generators of the spin group $SU(2)$, obey the same commutation relations. In fact, the groups $SO(3)$ and $SU(2)$ possess the same Lie algebra, meaning that their *local* structure is identical. However, the groups differ in their *global* properties. Rotations in 3D return to the identity after $2\pi$, but the rotation of a spin with Pauli matrices returns to the identity only after $4\pi$. In fact, the Lie algebra captures only the local structure of a group, but not its global topology. In fact, it is the global topology what makes $SO(3)$ and $SU(2)$ different.

As an analogy you may think of the famous Möbius strip. If you were living on the surface of the Möbius strip, it would just look like an ordinary $\mathbb{R}^2$, in particular, Taylor expansion around some point would not reveal something special. Nevertheless the global topology of a Möbius strip differs from that of a flat $\mathbb{R}^2$.

**The special orthochronous orthogonal group $SO^+(3,1)$ and its Lie algebra**

Let us now turn to the case of special orthochronous Lorentz transformations, that is, Lorentz transformation which preserve the direction of time and the orientation (handedness) in space. Clearly, the group of rotations $SO(3)$ discussed before is just a subgroup of $SO^+(3,1)$ in the spatial components. In addition, we need three generator for Lorentz boosts. Altogether we need six linearly independent generators which are denoted by $\lambda_{(\alpha\beta)}$. It turns out that these generators are given by the expression

$$[\lambda_{(\alpha\beta)}]_V^\mu = \delta_\alpha^\mu \eta_{\beta\nu} - \delta_\beta^\mu \eta_{\alpha\nu}$$  \hfill (2.90)

which is antisymmetric in the labels $\alpha, \beta \in 0, 1, 2, 3$. One can show that these generators obey the commutation relations

$$[\lambda_{(\alpha\beta)}, \lambda_{(\gamma\delta)}] = -\left( \eta_{\alpha\gamma} \lambda_{(\beta\delta)} - \eta_{\alpha\delta} \lambda_{(\beta\gamma)} + \eta_{\beta\gamma} \lambda_{(\alpha\delta)} - \eta_{\beta\delta} \lambda_{(\alpha\gamma)} \right)$$  \hfill (2.91)
Explicitly they are given by

\[
\begin{align*}
\lambda_{(01)} &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
\lambda_{(02)} &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
\lambda_{(03)} &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
\lambda_{(12)} &= \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \\
\lambda_{(13)} &= \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \\
\lambda_{(23)} &= \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}
\end{align*}
\]

Here all matrices have to be understood as the components of \([\lambda_{(\alpha\beta)}]^{\mu\nu}_\lambda\). The double index \((\alpha\beta)\) in round brackets indicates the two-dimensional plane in which the transformation takes place. As can be seen, the generators in the upper row generate the Lorentz boosts while the generators in the lower row generate ordinary rotations. Amazingly, Lorentz boosts and rotations appear here in a unified framework.

As we have six generators, the orthochronous special orthogonal group \(SO^+(3,1)\) is a 6-dimensional Lie group. The full transformations \(\Lambda\) can be obtained by applying the exponential function with six different control parameters \(\theta^{(\alpha\beta)}\):

\[
\Lambda = \exp\left( \sum_{0\leq\alpha<\beta\leq3} \theta^{(\alpha\beta)} \lambda^{(\alpha\beta)} \right)
\]

This exponential function can be computed since the Lie algebra closes, meaning that higher powers of the generators can be reduced to lower ones as follows:

\[
\begin{align*}
\text{Lorentz boosts:} & \quad \lambda^3_{(\beta\beta)} = \lambda_{(\beta\beta)} \quad \beta = 1 \ldots 3 \\
\text{Ordinary rotations:} & \quad \lambda^3_{(\alpha\beta)} = -\lambda_{(\alpha\beta)} \quad 1 \leq \alpha < \beta =\leq 3
\end{align*}
\]

Because of these relations the Taylor series of the exponential function involves only the generators and their squares but no higher powers. This allows us to decompose the exponential series into an even and an odd part (see exercises). For example, for \((\alpha\beta) = (1,2)\) one obtains

\[
\Lambda(\phi) = \exp(\phi\lambda_{(12)}) = 1 + \lambda^2_{(12)}(1 - \cos \phi) + \lambda_{(12)} \sin \phi,
\]

which is nothing but an ordinary rotation in the \(xy\)-plane:

\[
\Lambda(\phi) = \begin{pmatrix} 1 & \cos \phi & \sin \phi \\ -\sin \phi & \cos \phi & \cos \phi \\ \end{pmatrix}
\]

However, for \((\alpha\beta) = (1,0)\) we obtain

\[
\Lambda(\theta) = \exp(\theta\lambda_{(01)}) = 1 + \lambda^2_{(01)}(\cosh \theta - 1) + \lambda_{(01)} \sinh \theta,
\]
which is just a Lorentz boost in $x$-direction by the rapidity $\theta$:

$$\Lambda(\theta) = \begin{pmatrix}
\cosh \theta & \sinh \theta \\
\sinh \theta & \cosh \theta \\
1 & 1
\end{pmatrix}. \quad (2.98)$$

In many textbooks these six generators are split into two groups of three generators by defining

$$L_j := -\frac{i}{2} \sum_{k,l=1}^3 \epsilon_{jkl} \lambda_{(kl)}, \quad M_j := -i \lambda_{(0j)} \quad (j = 1, 2, 3). \quad (2.99)$$

Again the $L_j$ form the angular momentum vector while the $M_j$ generate the Lorentz boosts. The corresponding Lie algebra, known as the Lorentz algebra, reads

$$[L_i, L_j] = i \epsilon_{ijk} L_k \quad [L_i, M_j] = i \epsilon_{ijk} K_k \quad [M_i, M_j] = -i \epsilon_{ijk} L_k. \quad (2.100)$$

**Remark:** For the mathematical experts we would like to point out that the Lorentz algebra is reducible. In fact, if we define $S_{j}^{\pm} := I_j \pm iK_j$, then the commutation relations can be written in the form $[S_{j}^{\pm}, S_{j}^{\pm}] = i \epsilon_{ijk} S_{k}^{\pm}$ and $[S_{j}^{\pm}, S_{j}^{\mp}] = 0$. The latter relation tells us that the Lorentz algebra decomposes into two mutually commuting subalgebras $S_{j}^{+}$ and $S_{j}^{-}$, which are basically two copies of the group $SU(2)$ which are generated by Pauli matrices. The reducibility of the Lorentz algebra may be seen as the deeper reason for the existence of “spin” as a physical phenomenon.

### 2.2.3. Illustration of Lorentz boosts

In Fig. 2.5 you can see how a Lorentz boost in 1+1 dimensions

$$\begin{pmatrix} ct \\ x \end{pmatrix} = \begin{pmatrix} \cosh \theta & \sinh \theta \\
\sinh \theta & \cosh \theta \end{pmatrix} \begin{pmatrix} ct \\ \tilde{x} \end{pmatrix}$$

distorts an image. It is of course completely meaningless to draw a 2D image of the Würzburg residence in 1+1-dimensional spacetime, but the purpose of the figure is to visualize the main properties of a Lorentz boost, namely:

- The origin (the image center) remains invariant.
- Linearity: Straight lines remain straight.
- The light cone (orange lines) is mapped onto itself.
- The image is squeezed towards the diagonal. (In the limit $v \to c$ it will be flattened on the diagonal.)
2.2 Minkowski space

Figure 2.5.: Würzburg Residence in \((x, ct)\)-coordinates mapped by a Lorentz boost with rapidity \(\theta = 0.4\).

It is instructive to compare Lorentz boosts with ordinary rotations in the 2D plane. Ordinary rotations are given by the transformation law

\[
\begin{pmatrix}
x \\
y
\end{pmatrix} = \begin{pmatrix}
\cos \phi & \sin \phi \\
-\sin \phi & \cos \phi
\end{pmatrix}
\begin{pmatrix}
\tilde{x} \\
\tilde{y}
\end{pmatrix}.
\]

Rotations preserve the standard scalar product and therewith the Euclidean distance \(\ell^2 = (\Delta x)^2 + (\Delta y)^2\). Therefore, the points in the plane are transported on circles with a fixed radius \(r^2 = x^2 + y^2\). Note that the group of rotations (the \(SO(2)\)) is a compact one because after a rotation by \(\phi = 2\pi\) the original situation is recovered.

This has to be compared with the action of a Lorentz boost in 1+1 dimensions, as shown in the adjacent figure. A Lorentz boost preserves the pseudo scalar product and therewith the Minkowski distance \(\ell^2 = (\Delta x)^2 - c^2(\Delta t)^2\). Therefore, the points in the plane are transported along hyperbolas with \(x^2 - c^2 t^2 = \text{const}\). In contrast to rotations, where only the origin has distance zero, we have a light cone (orange lines) of points with distance zero from the origin which are mapped onto themselves. In addition, the transformation keeps the three sectors of the Minkowski space (future, past and spacelike region) separated, which is necessary to ensure causality.

Note that the points on the light cone are still transported inwards or outwards along the light cone. Moreover, in contrast to rotations, which are cyclic in \(2\pi\), the Lorentz...
boosts are *non-compact*: They never return to the identity.

**Time dilation and length contraction**

Having understood how a Lorentz boost maps the points in a 1+1-dimensional spacetime, it is now easy to understand and interpret the phenomena of length contraction and time dilation.

Consider a rod of length $\tilde{a}$ resting in the co-moving system $\tilde{S}$ (see figure). This rod moves upwards in time, sweeping out the vertical blue area shown in the figure. If we apply a Lorentz boost, this vertical strip becomes tilted and, as illustrated by the red bullet, its width decreases. Therefore, the observer in the laboratory system sees a reduced length $a < \tilde{a}$.

In order to understand the phenomenon of time dilation, consider a clock resting at the origin in $\tilde{S}$. As shown in the figure, the time span elapsed between the origin and the blue bullet is $c\tilde{\tau}$. The Lorentz boost shifts the blue bullet to a different time $\tau > \tilde{\tau}$. Hence the observer from the laboratory frame perceives the moving clock as slower compared to a resting one. This is in agreement with our previous findings (see Eq. (2.29) on page 31):

$$a = \frac{1}{\gamma} \tilde{a}, \quad \tau = \gamma \tilde{\tau}. \quad (2.101)$$

**Simultaneity and presence**

In Newton’s world, time is a global parameter. Events at different locations in spacetime are said to happen *simultaneously* if they take place at the same time. In the relativistic world, two events happening at the same time in a given coordinate system are no longer simultaneous in a different coordinate system. In fact, as one can see in Fig. 2.5, the horizontal red line in the Minkowski diagram (representing a multitude of simultaneous events), mapped by a Lorentz boost, results in a tilted line of non-simultaneous events.

The impossibility of defining simultaneous at different locations also implies that our common notion of presence (German: Gegenwart) has to be revised. Our common sense tells us that certain events around us are happening *now* in the present moment. In the relativistic setting, where the notion of simultaneity does no longer exist, there is no meaning of happening now except for events taking place exactly at the observers location.
Thinking more deeply about it, it turns out that the presence, seen from your own perspective, may be defined as a set of events with the following plausible properties:

- The presence cannot change your past.
- In the future you cannot change the present.

Confused? Convince yourself that the negations of these statements are true: Of course the presence can change your future, and of course you had in the past the opportunity to change the present.

Consequently the presence comprises all events in the space-like region of the light cone, and since the light cone itself is Lorentz-invariant, the above definition of the presence is also Lorentz-invariant. Thus, the presence is no longer sharply defined as a certain time slice, it rather consists of a whole range of events that could be made to take place simultaneously by choosing an appropriate reference frame.

From a different perspective, the notion of now is valid only in a certain spatial range depending on the required time resolution. On the scale of nanoseconds, the spatial range is as small as 30 cm. On a human scale of $10^{-1}$ seconds, the spatial range covers is 30,000 km, spanning the whole globe. This is the only reason why in our everyday world the concept of simultaneity and presence seems to work perfectly.
3. Relativistic Mechanics

3.1. Equations of Motion

3.1.1. Mass Points and World Lines

As in Newtonian Mechanics, the basic concept of Relativistic Mechanics (RM) relies on the notion of a mass point. A mass point is an idealized physical object which has a well-defined position but they have no size and no rotational degrees of freedom. It is assumed that the total mass of the object is concentrated in a single point, very much like a Dirac delta function. This concept of a mass point is basically the same as in Newtonian mechanics, but there are some important differences:

- In Minkowski space, even if the particle is at rest, it proceeds in time, hence the mass point moves on a line, the so-called world line.
- As we will see, the effective mass of the particle depends on its velocity and diverges as the particle velocity approaches the speed of light. Therefore, the mass $m$ always denotes the rest mass of the particle.

Parameterization of World Lines

Physically, a massive object can only move along a line of events that are causally connected, meaning that its velocity is smaller than $c$. Therefore, the world line of the mass point is a timelike curve which is confined to the interior of its light cone at any time, as sketched in the figure.

From the mathematical point of view, a worldline is a parameterized curve of the form

$$C : [a, b] \to \mathbb{R}^{3+1} : \lambda \mapsto x(\lambda) = \begin{pmatrix} x^0(\lambda) \\ x^1(\lambda) \\ x^2(\lambda) \\ x^3(\lambda) \end{pmatrix}$$

constrained by the condition that the tangent vec-
tor is timelike, i.e.
\[ \frac{dx^\mu(\lambda)}{d\lambda} \frac{dx_\mu(\lambda)}{d\lambda} < 0. \]

It is well-known that a given curve can be parameterized in many ways, differing essentially in the pace at which the parameter changes along the curve. Therefore, two different parameterizations \( C(\lambda) \) and \( \tilde{C}(\tau) \) describe the same curve if we can remap their parameters onto each other, that is, if we can find a strictly monotonously increasing function \( f : \mathbb{R} \to \mathbb{R} \) such that
\[ \tau = f(\lambda), \quad C(\lambda) = \tilde{C}(f(\lambda)). \quad (3.1) \]

Although the curves are identical, the corresponding tangent vectors differ by a factor of \( df / d\lambda \):
\[ \frac{dx^\mu(\lambda)}{d\lambda} = \frac{d\tilde{x}^\mu(\tau)}{d\tau} \frac{df(\lambda)}{d\lambda}. \quad (3.2) \]
This implies that the tangent vectors differ in their squared relativistic lengths by
\[ \left( \frac{dx^\mu(\lambda)}{d\lambda} \frac{dx_\mu(\lambda)}{d\lambda} \right) = \left( \frac{d\tilde{x}^\mu(\tau)}{d\tau} \frac{d\tilde{x}_\mu(\tau)}{d\tau} \right) \left( \frac{df(\lambda)}{d\lambda} \right)^2. \quad (3.3) \]

**Proper-time parameterization**

We can use the freedom of reparameterization to choose a special parameterization in such a way that
\[ \frac{d\tilde{x}^\mu(\tau)}{d\tau} \frac{d\tilde{x}_\mu(\tau)}{d\tau} = -c^2. \quad (3.4) \]

**Proof:** Suppose that we already have a worldline \( C(\lambda) \) in some arbitrary parameterization. Then Eq. (3.3) tells us that
\[ \frac{df(\lambda)}{d\lambda} = \frac{1}{c} \sqrt{-\frac{dx^\mu(\lambda)}{d\lambda} \frac{dx_\mu(\lambda)}{d\lambda}} \]

Hence the reparameterization function \( \tau = f(\lambda) \) is given (up to a constant) by
\[ f(\lambda) = \frac{1}{c} \int \sqrt{-\frac{dx^\mu(\lambda)}{d\lambda} \frac{dx_\mu(\lambda)}{d\lambda}} \ d\lambda. \quad \square \]

Therefore, without loss of generality, we can always choose a parameterization \( x(\tau) \) in such a way that
\[ \frac{dx^\mu}{d\tau} \frac{dx_\mu}{d\tau} = -c^2. \quad (3.5) \]

What is so special about this parameterization? To this end let us Lorentz-transform the coordinates in a given point in Minkowski space into the co-moving frame\(^2\) with coordinates \( \tilde{x}(\tau) \). Clearly, the scalar relation (3.5) remains invariant under this transfor-

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1. In the parlance of Theoretical Physics, this is like “fixing a gauge”.
2. Note that the worldline can be curved, so it may be accelerated. Nevertheless it is possible to define a local co-moving inertial frame in each point of the trajectory.
3.1 Equations of Motion

mation. But on the other hand we know that \( \ddot{x}^1 = \ddot{x}^2 = \ddot{x}^3 = 0 \), hence we obtain

\[
\frac{d\ddot{x}^\mu}{d\tau} \frac{d\ddot{x}_\mu}{d\tau} = -c^2 \left( \frac{d\ddot{\tau}}{d\tau} \right)^2 = -c^2 \Rightarrow \tau = \ddot{\tau}.
\] (3.6)

This demonstrates that \( \tau \) is the time that the point particle experiences by itself, the so-called proper time of eigen time (German: Eigenzeit) of the mass point.

The proper time \( \tau \) of an object is defined as the time displayed by a clock attached to the object.

From now on we will tacitly assume that worldlines of mass points, if not stated otherwise, are parameterized by their proper time \( \tau \). Derivatives with respect to the proper time are usually denoted by a dot, i.e.

\[
\dot{x} := \frac{dx}{d\tau}.
\] (3.7)

3.1.2. Relativistic 4-velocity and relativistic 4-momentum

Relativistic velocity

The relativistic velocity or 4-velocity of a mass point is defined by

\[
u := \dot{x} = \frac{dx}{d\tau}.
\] (3.8)

where the dot denotes the total derivative with respect to the proper time \( \tau \). Since \( \tau \) is a scalar that does not change under Lorentz transformation, and given that \( x \) is a 4-vector, it is plausible that \( \dot{x} \) has to be a 4-vector as well.

Unlike an ordinary velocity \( \vec{v} \), which can be zero, the magnitude of the 4-velocity is constant and essentially given by the speed of light:

\[
u^\mu \nu_{\mu} = -c^2.
\] (3.9)

Roughly interpreted, a relativistic object can never be at rest. It could rest in space, but then it has to propagate in time. In fact, a spatially resting particle has the 4-velocity

\[
\text{Resting particle: } \nu^\mu = \left( \frac{c}{\ddot{\tau}} \right).
\] (3.10)

If we apply a Lorentz transformation we get

\[
\nu^\mu = \left( \frac{\gamma c}{\gamma \ddot{\tau}} \right) = \frac{1}{\sqrt{1 - \ddot{\tau}^2}} \left( \begin{array}{c} c \\ v_x \\ v_y \\ v_z \end{array} \right),
\] (3.11)
where \( \gamma = 1 / \sqrt{1 - \beta^2 / c^2} \) is the Lorentz factor.

**Relativistic momentum**

The **relativistic momentum** or **4-momentum** is defined by

\[
\mathbf{p} := m \mathbf{u} = m \mathbf{x}
\]

(3.12)

where \( m \) is the rest mass of the particle. As \( m \) is by definition invariant under Lorentz transformations, it is clear that \( \mathbf{p} \) transforms like a 4-vector. The components of this vector are interpreted as the **relativistic energy** divided by \( c \) in the zeroth component and the **relativistic 3-momentum** \( \mathbf{p} \) in the spatial components:

\[
p^\mu = \left( \frac{E}{c}, \mathbf{p} \right)
\]

(3.13)

With Eq. (3.11) these quantities are given by

\[
E = \gamma m c^2, \quad \mathbf{p} = \gamma m \mathbf{v}.
\]

(3.14)

These definitions of energy and momentum will become clear below when discussing the Lagrange formalism of a relativistic particle. Notably, for a resting particle with \( \mathbf{v} = 0 \) the first relation reduces to the world-famous formula

\[
E = mc^2
\]

(3.15)

Since \( \mathbf{p} = m \mathbf{u} \) and \( u^\mu u_\mu = -c^2 \) we can immediately conclude that the relativistic momentum 4-vector has a constant length:

\[
p^\mu p_\mu = m^2 u^\mu u_\mu = -m^2 c^2.
\]

(3.16)

Inserting (3.13) this gives the relativistic energy-momentum relation

\[
E^2 = m^2 c^4 + \mathbf{p}^2 c^2.
\]

(3.17)

Solving for \( E \) this expression can be Taylor-expanded in the momentum, giving:

\[
E = \pm \left( mc^2 + \frac{p^2}{2m} - \frac{p^4}{8m^3c^2} + O(p^6) \right)
\]

(3.18)

This simple expansion is interesting on its own. On the one hand, taking the square root we get two solutions with different sign. As we will see, the existence of two solutions is in some sense related to the existence of antimatter. On the other hand, we have a zeroth order contribution, namely, a constant offset, as expressed by the famous formula \( E = mc^2 \). This contribution tells us that a massive particle carries a rest energy, or more philosophically, that mass is some kind of frozen energy. Then the second-order contribution, the only one where \( c \) does not enter, is the usual non-relativistic kinetic energy \( E_{\text{kin}} = \frac{1}{2} m \mathbf{v}^2 = \frac{\mathbf{p}^2}{2m} \) in Newtonian mechanics. Then the fourth-order term is the first relativistic correction.
3.1 Equations of Motion

**4-acceleration**

Finally, we can take the derivative of the 4-velocity with respect to the proper time to define a 4-acceleration

\[ a := \dot{u} = \ddot{x} = \frac{d^2x}{d\tau^2} \]  

(3.19)

We can show (see exercise) that the components of the 4-acceleration are given by

\[ a = \left( \frac{\gamma \dot{c}}{\gamma^2 \tilde{t} + \gamma \tilde{c}} \right). \]  

(3.20)

### 3.1.3. Action

In non-relativistic classical mechanics we have learned that the equations of motion can be derived from the principle of least action.

**Recall:** The action of a point particle with mass \( m \) moving along the trajectory \( q(t) \) in the potential \( V(q) \) is given by

\[ S[q] = \int_{t_1}^{t_2} L(q, \dot{q}) \, dt \quad \text{where} \quad L = T - V = \frac{1}{2} m \dot{q}^2 - V(q) \]  

(3.21)

The principle of least action tells us that the trajectory, which is realized in nature, is the one for which the action is extremal (minimal). Consequently, a variation of the trajectory (keeping the ending points fixed) should not change the action to lowest order:

\[ 0 = \delta S = \int_{t_1}^{t_2} \left( \frac{\partial L(q, \dot{q})}{\partial q} \delta q + \frac{\partial L(q, \dot{q})}{\partial \dot{q}} \delta \dot{q} \right) \]  

(3.22)

In this expression and variations \( \delta q \) and \( \delta \dot{q} \) are not independent because one is the derivative of the other. This difficulty can be circumvented by partial integration. Taking into account that the variation vanishes at the boundaries, this gives the Lagrange equation of motion:

\[ 0 = \delta S = \int_{t_1}^{t_2} \left( \frac{\partial L(q, \dot{q})}{\partial q} - \frac{d}{dt} \frac{\partial L(q, \dot{q})}{\partial \dot{q}} \right) \delta q \Rightarrow \frac{\partial L}{\partial q} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} = 0. \]  

(3.23)

In classical mechanics the principle of least action is introduced as an axiom. Later this principle can be justified in retrospect in the framework of quantum mechanics. In quantum mechanics we are dealing with a superposition of classical configurations, each of them weighted by a complex phase factor \( e^{iS/\hbar} \). Due to interference most of these configurations cancel each other except those for which the action is almost stationary. This happens roughly along the classical trajectories in a corridor with an uncertainty of the order of \( \hbar \). This highlights that the action has an absolute meaning independent of the choice of the coordinates. In our mathematical framework this means that

**The action \( S \) is always a scalar quantity which is invariant under Lorentz transformations.**

This gives us already a clue how to define the action properly in the relativistic case. Another clue comes from the so-called **principle of heuristic simplicity**. This principle tells us that nature is not unnecessarily complicated. With respect to the action this
means that we should always choose the simplest nontrivial scalar expression which is compatible with the symmetries of the system under consideration.

**Relativistic action of a free point particle**

As we have seen above, the point particle moves along a worldline. So a natural candidate would be the relativistic length $\int_C \, ds$ of the worldline $C$ between two given events, say $A$ and $B$. In addition, the action needs an appropriate coupling constant, for which the mass $m$ is a natural candidate. Furthermore, the action has to have the dimension of an action ($m^2 \text{kg/s}$ in SI units), which explains why we need another factor $c$. Finally, the physical solution should correspond to the least action, requiring a minus sign in front. Altogether the simplest candidate for the action of a point particle is given by

$$S = -mc \int_C \, ds$$

where $ds = \sqrt{-dx^\mu dx_\mu}$ is the relativistic timelike distance along the trajectory. Because of $ds = c \, d\tau$ the action is just proportional to the integrated proper time $\tau$ elapsed along the trajectory. This suggests that we may also write

$$S = -mc^2 \int_C \, d\tau. \quad (3.25)$$

But there is an important caveat in using the proper time. The equation (3.25) is of course formally correct, but we cannot simply use fixed integral boundaries such as $\int_{T_A}^{T_B} \, d\tau$ in the variational machinery, simply because each trajectory needs a different proper time so that the boundary values of $\tau$ would depend on the choice of the trajectory. But in order to compute the integral, however, we need fixed boundary values.\(^3\)

One way to circumvent this problem is to parameterize all trajectories over which we want to vary by a parameter $\lambda$ in such a way that its value at the ending points is the same for all trajectories, say $\lambda_A$ and $\lambda_B$. Then we can write

$$S = -mc \int_C ds = -mc \int_{\lambda_A}^{\lambda_B} ds \, d\lambda = -mc \int_{\lambda_A}^{\lambda_B} \sqrt{-\frac{dx^\mu}{d\lambda} \frac{dx_\mu}{d\lambda}} \, d\lambda. \quad (3.26)$$

In analogy to what we have learned in mechanics, we may therefore write the action as

$$S = \int_{\lambda_A}^{\lambda_B} L(x, \dot{x}) \, d\lambda, \quad (3.27)$$

where

$$L(x, \dot{x}) = -mc \sqrt{-\eta_{\mu\nu} \dot{x}^\mu \dot{x}^\nu} \quad (3.28)$$

\(^3\)This is a very simple example demonstrating that one must not fix the gauge (the parameterization) in the action before the variation is carried out. Keep this in mind because similar problems occur frequently in advanced quantum field theory.
is the Lagrange function of a free massive point particle which in this case depends only on the velocity $\dot{x}^\mu = \frac{dx^\mu}{d\lambda}$ with respect to the parameter $\lambda$.

**Equations of motion**

We can easily compute the equations of motion in the same way as in nonrelativistic mechanics. The variation of the action is given by

$$\delta S = \int_{\lambda_A}^{\lambda_B} \left( \frac{\partial L(x, \dot{x})}{\partial x^\mu} \delta x^\mu + \frac{\partial L(x, \dot{x})}{\partial \dot{x}^\mu} \delta \dot{x}^\mu \right) d\lambda$$

(3.29)

and we would like to find a solution for which the variation vanishes to first order, that is, $\delta S = 0$. Here the problem arises that the variations $\delta x^\mu$ and $\delta \dot{x}^\mu$ are not independent. However, as in the case of classical mechanics, this problem can be circumvented by partially integrating the second term:

$$0 = \delta S = \left[ \frac{\partial L(x, \dot{x})}{\partial \dot{x}^\mu} \delta \dot{x}^\mu \right]_{\lambda_A}^{\lambda_B} + \int_{\lambda_A}^{\lambda_B} \left( \frac{\partial L(x, \dot{x})}{\partial x^\mu} - \frac{d}{d\lambda} \frac{\partial L(x, \dot{x})}{\partial \dot{x}^\mu} \right) \delta x^\mu d\lambda$$

(3.30)

Since the variation of the trajectory vanishes at the ending points, the boundary term in square brackets vanishes. Moreover, the variation $\delta x^\mu$ is arbitrary, hence the integral can only vanish if and only if the integrand in round brackets vanishes. This leads us directly to the relativistic Euler-Lagrange equations of motion

$$\frac{\partial L(x, \dot{x})}{\partial x^\mu} - \frac{d}{d\lambda} \frac{\partial L(x, \dot{x})}{\partial \dot{x}^\mu} = 0.$$  

(3.31)

**Canonically conjugated 4-momentum**

The relativistic canonical momentum is defined as

$$p^\mu = \frac{\partial L}{\partial \dot{x}^\mu}, \quad p_\mu = \frac{\partial L}{\partial x^\mu}.$$  

(3.32)

This allows us to rewrite the Lagrange equations of motion as

$$p^\mu = \frac{\partial L}{\partial \dot{x}^\mu},$$

(3.33)

where the right hand side may be interpreted as a relativistic force.
**Free particle**

In the case of a free particle, where the Lagrange function is given by Eq. \((3.28)\), the canonically conjugated 4-momentum is given by

\[
p_\rho = -mc \frac{\partial \sqrt{-\eta_{\mu\nu} \dot{x}^\mu \dot{x}^\nu}}{\partial \dot{x}^\rho} = \frac{mc}{2 \sqrt{-\eta_{\mu\nu} \dot{x}^\mu \dot{x}^\nu}} \left( \eta_{\mu\nu} \dot{x}^\nu + \eta_{\mu\nu} \dot{x}^\nu \right) = \frac{mc \dot{x}_\rho}{\sqrt{-\eta_{\mu\nu} \dot{x}^\mu \dot{x}^\nu}} (3.34)
\]

and likewise

\[
p^\rho = mc \dot{x}^\rho. (3.35)
\]

**Choosing the parameterization: Gauge fixing**

What is the origin of the strange square root in the denominator? To answer this question, recall that we are working with an arbitrary parameterization of the trajectories which are only constrained to take particular values \(\lambda_A\) and \(\lambda_B\) at the ending points. What happens if we change the parameterization? Let us e.g. consider the map

\[
\lambda \mapsto \tilde{\lambda} = f(\lambda) (3.36)
\]

where \(f(\lambda)\) is a monotonously increasing function. Then the derivative changes as

\[
\frac{d}{d\lambda} \mapsto \frac{d}{d\tilde{\lambda}} = \frac{1}{f'(\lambda)} \frac{d}{d\lambda}. (3.37)
\]

First we note that the Lagrange function is *not* invariant under reparameterization:

\[
L \mapsto \tilde{L} = -mc \sqrt{-\eta_{\mu\nu} \dot{\tilde{x}}^\mu \dot{\tilde{x}}^\nu} = -\frac{mc}{|f'(\lambda)|} \sqrt{-\eta_{\mu\nu} \dot{x}^\mu \dot{x}^\nu}. (3.38)
\]

However, in the conjugated momentum

\[
p^\rho = \frac{\partial L}{\partial \dot{x}_\rho} = \frac{mc \dot{x}_\rho}{\sqrt{-\eta_{\mu\nu} \dot{x}^\mu \dot{x}^\nu}} (3.39)
\]

the divisor \(f'(\lambda)\) cancels out because \(\frac{d}{d\lambda}\) occurs both in the nominator and the denominator. This means that the 4-momentum has a parameterization-independent (gauge-invariant) meaning; it is always the same, no matter which parameterization we choose.

To make life easier, we may therefore choose a particular parameterization (fix the gauge) in a way that the formulas become as simple as possible. Obviously, the convenient choice is to take the proper time \(\lambda = \tau\). In the case the 4-velocity has the square norm \(\eta_{\mu\nu} \dot{x}^\mu \dot{x}^\nu = -c^2\), reducing the definition of the 4-momentum to

\[
p^\mu = m \dot{x}^\mu = mu^\mu (3.40)
\]

or in short

\[
\mathbf{p} = mu. (3.41)
\]

This is in fact what we naively expect: Momentum is mass times velocity.
3.1 Equations of Motion

We can relate the 4-momentum of a free particle to the ordinary velocity \( \vec{v} = \frac{d}{dt}\vec{x} \) by

\[
p = mu = \left( \gamma mc \right) \left( \gamma mv \right).
\]  

(3.41)

Please note again that the choice of the particular parameterization (gauge fixing) is allowed only \textit{after} deriving the equations of motion since otherwise the variation of the trajectory would violate the constraints at the boundaries.

\textbf{Relativistic 4-force}

The concept of a \textit{relativistic force} is straightforward, but there are surprises. As expected, the relativistic force is defined as the derivative of the conjugate momentum with respect to the proper time:

\[
f = \frac{d}{d\tau} p = \dot{p} = ma.
\]  

(3.42)

In components the 4-force is given by

\[
f = \left( \gamma \frac{dE}{d\tau} \gamma \vec{F} \right),
\]  

(3.43)

where \( \vec{F} \) is the usual Newton-type 3-force. Now imagine that a resting point particle with mass \( m \) is subjected to a force in which only the zeroth component does not vanish:

\[
p(\tau = 0) = \begin{pmatrix} mc \\ 0 \\ 0 \\ 0 \end{pmatrix}, 
\quad f = \begin{pmatrix} a \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad p(\tau) = \begin{pmatrix} mc + a\tau \\ 0 \\ 0 \\ 0 \end{pmatrix}
\]  

(3.44)

which would mean that the particle is behaving like someone eating fast food: It is not accelerating at all, it remains at rest but it is gaining weight: \( m(\tau) = m(0) + \tau a/c \).

Isn’t that strange? We apply a constant force and all what happens is that the object gains weight? But in fact, this is not as exotic as you might think. Actually this happens every day when you charge your smartphone. The battery absorbs an energy \( \Delta E \approx 10\text{Wh} = 36000\text{ J} \). Because of \( E = mc^2 \) your device gains the weight \( \Delta m = \Delta E/c^2 \approx 4 \cdot 10^{-13}\text{kg} \).

What we learn from this example is that the relativistic force has a zeroth component which encoded the influx or loss of energy without changing the velocity. This inevitably changes the mass of the particle.

If we are interested in “ordinary” forces that do not change the rest mass, we have to ensure that \( ||p||^2 = p_\mu p^\mu = -m^2c^2 \) is preserved in time, meaning that \( \frac{d}{d\tau} p_\mu p^\mu = 2p_\mu \dot{p}^\mu = 0 \). A mass-preserving 4-force is therefore (relativistically) orthogonal on the 4-velocity:

\[
f \cdot u = f_\mu u^\mu = 0.
\]  

(3.45)
In the representation (3.43), such a mass-conserving force has the form
\[
\mathbf{f} = \left( \frac{2}{c} \mathbf{F} \cdot \mathbf{\dot{v}} \right),
\]
(3.46)
where \( \frac{dE}{dt} = \mathbf{F} \cdot \mathbf{v} \) is just Newton’s “force times distance” law, describing how much work is performed on the particle per time.

**Remark:** From Newtonian mechanics we are familiar with the concept that the 3-force \( \mathbf{F} \) is the same in all inertial frames, meaning that it is an invariant under Galilei transformations. Not so in Special Relativity. Here the 4-force \( \mathbf{f} \) is an ordinary 4-vector which transforms under Lorentz transformations like any other 4-vector. One can show that generally such a transformation \( x \rightarrow \tilde{x} \) also changes the corresponding 3-force \( \mathbf{f} \) which can be computed via Eq. (3.46), i.e. in general we have \( \mathbf{f} \neq \tilde{\mathbf{f}} \). The only exception is a Lorentz boost in the same direction as \( \mathbf{f} \). Then one can show that \( \mathbf{f} = \tilde{\mathbf{f}} \).

### 3.1.4. Charged particle in an electromagnetic field

#### The electromagnetic four-potential

An electromagnetic field is described by the electric potential \( \phi(\mathbf{x}, t) \) and the magnetic vector potential \( \mathbf{A}(\mathbf{x}, t) \). Using SI units, we can derive the physical electric field \( \mathbf{E} \) and the magnetic field \( \mathbf{B} \) from these potentials via
\[
\begin{align*}
\mathbf{E}(\mathbf{x}, t) &= -\nabla \phi(\mathbf{x}, t) - \frac{\partial \mathbf{A}(\mathbf{x}, t)}{\partial t}, \\
\mathbf{B}(\mathbf{x}, t) &= \nabla \times \mathbf{A}(\mathbf{x}, t).
\end{align*}
\]
(3.47)
As we will see below, the fields \( \phi \) and \( \mathbf{A} \) are not just a mathematical trick but they also reflect a physical reality; they play a fundamental role in electrodynamics.

In Special Relativity, the electric potential \( \phi \) and the magnetic vector potential \( \mathbf{A} \) are combined in a single four-vector \( \mathbf{A}(\mathbf{x}) \):
\[
\mathbf{A}^\mu(\mathbf{x}) = \begin{pmatrix} \frac{1}{c} \phi(\mathbf{x}, t) \\ \mathbf{A}(\mathbf{x}, t) \end{pmatrix}.
\]
(3.48)

**Action for a charged particle in a given electromagnetic field**

In order to find out how a charged relativistic particle moves in a given electromagnetic field, we have to extend the action (3.28) by an additional term. This term should be a scalar; it should not depend on the explicit coordinates \( \mathbf{x} \) but only on the four-velocity \( \mathbf{\dot{x}} \) and the four-field \( \mathbf{A}(\mathbf{x}) \). The simplest extension would be to add a term in which we

---

\(^4\)In some textbooks you will find instead \( \mathbf{E}(\mathbf{x}, t) = -\nabla \phi(\mathbf{x}, t) - \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} \) with an additional factor \( c^{-1} \) in front of the vector potential. These books use Gaussian units. For details see Appendix B on page 194.
forms a scalar out of these quantities:

\[ L(x, \dot{x}) = -mc \sqrt{-\eta_{\mu\nu} \dot{x}^\mu \dot{x}^\nu} + eA_\nu \dot{x}^\nu. \]  

(3.49)

Here \( e \) is a coupling constant, which in the end turns out to be the electric charge of the particle. Note that this Lagrangian does not include the dynamics of the electromagnetic field itself, so it does not yet give us the Maxwell equations. Instead, \( A(x) \) is just considered as an externally given electromagnetic field.

With the extended Lagrangian, let us now compute the Lagrange equations of motion (3.31):

\[ \frac{\partial L(x, \dot{x})}{\partial x^\mu} - d \frac{d \partial L(x, \dot{x})}{d\tau \partial \dot{x}^\mu} = 0. \]  

(3.50)

The first term gives

\[ \frac{\partial L(x, \dot{x})}{\partial x^\mu} = e \frac{\partial A_\nu}{\partial x^\mu} \dot{x}^\nu = e(\partial_\mu A_\nu) \dot{x}^\nu. \]  

(3.51)

In the second term, the conjugate momentum is given by

\[ \frac{\partial L(x, \dot{x})}{\partial \dot{x}^\mu} = m\dot{x}_\mu + eA_\mu(x(\tau)) \]  

(3.52)

and since \( \frac{d}{d\tau}A_\mu(x(\tau)) = \frac{\partial A_\mu}{\partial x^\nu} \frac{dx^\nu}{d\tau} \) we get

\[ \frac{d}{d\tau} \frac{\partial L(x, \dot{x})}{\partial \dot{x}^\mu} = m\ddot{x}_\mu + e(\partial_\nu A_\mu) \dot{x}^\nu. \]  

(3.53)

Summarizing these contributions we find that

\[ m\ddot{x}_\mu = e(\partial_\mu A_\nu - \partial_\nu A_\mu) \dot{x}^\nu. \]  

(3.54)

Note that we obtain an antisymmetric tensor in the round brackets. As we will see below, this tensor encodes the electric and the magnetic fields.

**The electromagnetic tensor**

Defining the electromagnetic tensor by

\[ F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \]  

(3.55)

the equation of motion can be written in the compact form

\[ m\ddot{x}_\mu = eF_{\mu\nu} \dot{x}^\nu. \]  

(3.56)

Obviously, the tensor \( F_{\mu\nu} \) is antisymmetric. In order to understand its components, let us consider the non-relativistic limit \( v \ll c \), where \( \frac{d}{d\tau} \approx \frac{d}{d\tau} \), and let us consider the
spatial components $\mu = i = 1, 2, 3$. Then
\[
m \ddot{x}_i = e \left( F_{0i} c + F_{ij} v^j \right).
\] (3.57)

If we compare this expression with the formula for the Lorentz force,
\[
m \vec{a} = e \left( \vec{E} + \vec{v} \times \vec{B} \right),
\] (3.58)
we are led to
\[
F_{\mu \nu} = \begin{pmatrix}
0 & -E_x/c & -E_y/c & -E_z/c \\
E_x/c & 0 & B_z & -B_y \\
E_y/c & -B_z & 0 & B_x \\
E_z/c & B_y & -B_x & 0
\end{pmatrix}
\] (3.59)
and similarly
\[
F^{\mu \nu} = \begin{pmatrix}
0 & E_x/c & E_y/c & E_z/c \\
-E_x/c & 0 & B_z & -B_y \\
-E_y/c & -B_z & 0 & B_x \\
-E_z/c & B_y & -B_x & 0
\end{pmatrix}
\] (3.60)

Note: As summarized in Appendix B on page 194, the electrodynamic quantities have the following SI units:
\[
[A] = \text{kg m s}^{-2}, \quad [F] = [\vec{B}] = \text{kg m s}^{-2}, \quad [\vec{E}] = \text{kg m s}^{-3}, \quad [e] = \text{As},
\]
where $A$ stands for the technical unit Ampere for electric currents. One can easily check that these units are consistent with the equation of motion (3.56).

**Gauge invariance**

One of the most important properties of electrodynamics is gauge invariance. In this lecture we will learn a lot about the origin of gauge invariance, its interpretation and consequences. At this point let us have a first glance.

We encountered the term *gauge* (german: *Eichung*) already in the context of the parameterization of a curve. Basically the term *gauge* stands for some artificial non-physical freedom in our formulas, a kind of redundancy. We can get from one gauge to another by *gauge transformations*, and we are free to select a particular gauge, e.g. with the aim to simplify the formulas. It is a good advice to think of a gauge as the freedom of parameterization or representation of a problem.

A gauge is an non-physical redundancy in our description of physical reality.

There is a fundamental gauge freedom in the Lagrange formalism itself. For example, let $L(x, \dot{x})$ be the Lagrange function of the relativistic point particle, where the dot denotes the total derivative with respect to the the curve parameter $\lambda$. If we transform

\[\footnote{These expressions are valid for the ‘mostly plus’ convention. For the ‘mostly minus’ convention the electric fields in the matrices have the opposite sign.} \]
the Lagrange function by adding a total derivative of the form
\[ L(x, \dot{x}) \rightarrow \tilde{L}(x, \dot{x}) = L(x, \dot{x}) + \frac{df}{d\lambda} \tag{3.61} \]
the corresponding action transforms as
\[ S \rightarrow \tilde{S} = \int_{\lambda_A}^{\lambda_B} \left( L(\lambda), \dot{x}(\lambda) \right) + \frac{df}{d\lambda} \right) = S + f(\lambda_B) - f(\lambda_A). \tag{3.62} \]
Since the variation is carried out by keeping the ending points fixed, the additional contribution is constant and drops out in the variation problem: \( \delta S = \delta \tilde{S} \). If the function \( f \) depends only on the position in spacetime, i.e., \( f(\lambda) \equiv f(x(\lambda)) \), the total derivative in (3.61) has to be formed by using the chain rule:
\[ \frac{df(x(\lambda))}{d\lambda} = \frac{\partial f(x)}{\partial x^\mu} \frac{dx^\mu}{d\lambda} = (\partial_\mu f) \dot{x}^\mu. \tag{3.63} \]
Remarkably, if we compare this expression with the action (3.49),
\[ L(x, \dot{x}) = -mc\sqrt{-\eta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu} + eA_\mu \dot{x}^\mu \tag{3.64} \]
then we realize that any gauge transformation of the Lagrangian of the form (3.61) can be absorbed in a redefinition of the 4-vector potential \( A_\mu \rightarrow A_\mu + \frac{1}{e} \partial_\mu f \). Conversely, any such change of the electromagnetic 4-potential does not change the trajectory of the particle. Hence, absorbing the pre-factor \( \frac{1}{e} \) into the function \( f \), we can conclude that the equations of motion are invariant under the electromagnetic gauge transformation
\[ A_\mu \rightarrow \tilde{A}_\mu = A_\mu + \partial_\mu f. \tag{3.65} \]
In fact, one can check easily that the electromagnetic tensor \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \) is indeed invariant under gauge transformations:
\[ F_{\mu\nu} \rightarrow \tilde{F}_{\mu\nu} = \partial_\mu A_\nu + \partial_\nu \partial_\mu f - \partial_\nu A_\mu - \partial_\mu \partial_\nu f = F_{\mu\nu} \tag{3.66} \]
and so it is immediately clear that the equations of motion \( m\ddot{x}_\mu = e F_{\mu\nu} \dot{x}^\nu \) do not feel the gauge transformation.

### 3.2. Noether theorem

A paradigmatic milestone in Theoretical Mechanics is the Noether theorem. Emmy Noether, born in Erlangen, was one of the first female mathematicians who entered an academic career, only a few years after the state of Bavaria opened the universities for women. Since at that time women were not allowed to become a professor, the faculty asked the Prussian minister for an exception:

„Eure Exzellenz bittet die mathematisch-naturwissenschaftliche Abteilung der philosophischen Fakultät der Göttingen Uni-
And the minister replied:

„Die Zulassung von Frauen zur Habilitation als Privatdozent begegnet in akademischen Kreisen nach wie vor erheblichen Bedenken. Da die Frage nur grundsätzlich entschieden werden kann, vermag ich auch die Zulassung von Ausnahmen nicht zu genehmigen, selbst wenn im Einzelfall dadurch gewisse Härten unvermeidbar sind...“

Roughly speaking, the Noether theorem tells us that a continuous symmetry is always connected with a conserved quantity, and it gives a clear-cut recipe how this conserved quantity can be calculated. In fact, the Noether theorem is one of the most powerful tools in classical mechanics and classical field theory.

3.2.1. Relativistic Noether theorem for particles

Recall nonrelativistic Noether theorem

Before we start with the relativistic version of the Noether theorem, let us briefly recall its textbook derivation in the non-relativistic case:

Recall: In nonrelativistic classical mechanics, one usually considers a continuous transformation of the coordinates controlled by a parameter \( s \in \mathbb{R} \), where \( s = 0 \) corresponds to the identity. The transformation maps a given trajectory continuously \( q(t) \) into another one, denoted as \( q_s(t) \). This transformation is a symmetry transformation if it minimizes the action for any \( s \) in the vicinity of \( s = 0 \). This means that the corresponding Lagrange functions can differ at most by the total derivative of some function \( \Lambda \) with respect to time (because this difference shows up in the action integral only as a boundary contribution):

\[
\frac{\partial}{\partial s} L(q_s, \dot{q}_s) \bigg|_{s=0} = \frac{d}{dt} \Lambda.
\] (3.67)

On the left side we can carry out the derivative via chain rule:

\[
\frac{\partial}{\partial q_s} L(q_s, \dot{q}_s) \frac{\partial q_s}{\partial s} + \frac{\partial}{\partial \dot{q}_s} L(q_s, \dot{q}_s) \frac{\partial \dot{q}_s}{\partial s}.
\] (3.68)

In this expression, let us insert a zero:

\[
\frac{\partial}{\partial s} L(q_s, \dot{q}_s) = \frac{\partial}{\partial q_s} L(q_s, \dot{q}_s) \frac{\partial q_s}{\partial s} - \frac{d}{dt} \left( \frac{\partial}{\partial q_s} L(q, \dot{q}_s) \right) \frac{\partial q_s}{\partial s} + \frac{d}{dt} \left( \frac{\partial}{\partial \dot{q}_s} L(q, \dot{q}_s) \right) \frac{\partial \dot{q}_s}{\partial s} + \frac{\partial}{\partial \dot{q}_s} L(q_s, \dot{q}_s) \frac{\partial \dot{q}_s}{\partial s} \bigg|_{s=0} = 0
\]

Here we can use the Lagrange equations of motion to see that the first two terms cancel. The remaining two terms (the third and the fourth one) can be written as the total time derivative of a single term by means of the product rule:

\[
\frac{\partial}{\partial s} L(q_s, \dot{q}_s) = \frac{d}{dt} \left( \frac{\partial}{\partial q_s} L(q, \dot{q}_s) \frac{\partial q_s}{\partial s} \right).
\] (3.69)

This allows us to rewrite Eq. (3.67) as

\[
\frac{d}{dt} \left( \frac{\partial}{\partial q_s} L(q, \dot{q}_s) \frac{\partial q_s}{\partial s} - \Lambda \right) \bigg|_{s=0} = \frac{d}{dt} \left( \frac{\partial}{\partial q_s} L(q, \dot{q}_s) \frac{\partial q_s}{\partial s} \bigg|_{s=0} - \Lambda \right) = 0
\] (3.70)
implying that the so-called Noether charge

\[ Q(t) = \left. p \frac{\partial q_s(t)}{\partial s} \right|_{s=0} - \Lambda(t) \]  

(3.71)
is conserved in time.

The derivation in the relativistic case works exactly in the same way. Again we consider a continuous transformation by infinitesimally deforming the trajectory \( x(\tau) \rightarrow x_s(\tau) \) with some small parameter \( s \in \mathbb{R} \), assuming that there is no deformation for \( s = 0 \). This infinitesimal transformation is called symmetry transformation if it minimizes the action independent of \( s \) (to lowest order), i.e., if all trajectories are valid classical solutions. This means that it changes the Lagrange function by at most a total proper-time derivative of some scalar function \( \Lambda \):

\[ \left. \frac{\partial}{\partial s} L(x_s(\tau), \dot{x}_s(\tau)) \right|_{s=0} = \frac{d \Lambda(\tau)}{d \tau}. \]  

(3.72)

Following exactly the same steps as in the non-relativistic case, this equation can be rewritten as

\[ \left. \frac{d}{d \tau} \left( \frac{\partial L(x_s, \dot{x}_s)}{\partial \dot{x}_s} \right) - \Lambda \right|_{s=0} = \left. \frac{d}{d \tau} \left( \frac{\partial L(x, \dot{x})}{\partial \dot{x}} \right) \right|_{s=0} - \Lambda = 0. \]  

(3.73)

Thus we can conclude that the Noether charge

\[ Q(\tau) = \left. \frac{\partial L(x_s, \dot{x}_s)}{\partial \dot{x}_s} \right|_{s=0} - \Lambda(\tau) \]  

(3.74)
is conserved along the trajectory. Therefore, the practical task in applying the Noether theorem consist of finding \( \Lambda(\tau) \), everything else follows automatically.

### 3.2.2. Applications of the Noether theorem

**Translations**

As an example let us consider an infinitesimal translation in the direction of the 4-vector \( f \)

\[ x \rightarrow x_s = x + sf \quad \Rightarrow \quad x^\mu \rightarrow x^\mu_s = x^\mu + sf^\mu. \]  

(3.75)

This includes translations in space as well as translations in time. Since the displacement \( \delta x = sf \) is constant, the derivative \( \dot{x} \) is unchanged, i.e.

\[ \dot{x}_s = \dot{x}. \]

Let us first check whether this infinitesimal transformation is a symmetry transformation. To this end we would like to find out how the Lagrange function varies under the
transformation. For a free particle (no potential) we find:

\[
\partial_s L = \lim_{s \to 0} \frac{L(x_s, \dot{x}_s) - L(x, \dot{x})}{s} = \frac{\partial L}{\partial \dot{x}^\mu} \frac{\partial x^\mu}{\partial s} + \frac{\partial L}{\partial \ddot{x}^\mu} \frac{\partial \dot{x}^\mu}{\partial s} = 0. 
\] (3.76)

Hence the Lagrangian does not change at all, meaning that translations are (as expected) symmetry transformations with constant \( \Lambda \) which can be set to zero. The corresponding conserved Noether charge reads

\[
Q = \left. \frac{\partial L(x_s, \dot{x}_s)}{\partial \dot{x}^\mu} \frac{\partial x^\mu}{\partial s} \right|_{s=0} = p_\mu f^\mu. 
\] (3.77)

This charge depends on the direction \( f \) in which the translation takes place. We may set it to one of the four basis vectors \( f := e_\nu \). Thus we get four independent charges:

\[
Q_\nu = p_\mu (e_\nu)^\mu = p_\mu \delta_\mu^\nu = p_\nu. 
\] (3.78)

Hence in the case of translations the corresponding Noether charge that is conserved along the trajectory is simply the 4-momentum. This is of course what we expect.

**Rotations and Lorentz boosts**

Let us now investigate infinitesimal rotations and Lorentz boosts. Here the transformation reads

\[
x \to x_s = (1 + s \lambda_{(\alpha\beta)}) x, 
\] (3.79)

where the \( \lambda_{(\alpha\beta)} \) are the generators of \( SO^+(3,1) \) defined in Eqs. (2.90)-(2.92). In components this transformation applies to both the coordinates and the derivatives:

\[
\begin{align*}
\dot{x}^\mu & \to \dot{x}_s^\mu = (\delta^\mu_\nu + s [\lambda_{(\alpha\beta)}]^\mu_\nu) \dot{x}^\nu \\
\ddot{x}^\mu & \to \ddot{x}_s^\mu = (\delta^\mu_\nu + s [\lambda_{(\alpha\beta)}]^\mu_\nu) \ddot{x}^\nu
\end{align*} 
\] (3.80)

Under this transformation the Lagrange function changes as

\[
\partial_s L = \left. \frac{\partial L}{\partial \dot{x}^\mu} \frac{\partial x^\mu}{\partial s} + \frac{\partial L}{\partial \ddot{x}^\mu} \frac{\partial \dot{x}^\mu}{\partial s} \right|_{s=0} = m \ddot{x}_\mu. 
\] (3.81)

One can easily show that this expression vanishes. To this end we recall the generators \( \lambda_{(\alpha\beta)} \) in Eq. (2.90) on page 44. The matrices \([\lambda_{(\alpha\beta)}]^\mu_\nu\) are partly symmetric (Lorentz boosts) and antisymmetric (rotations), but if we raise or lower one of the indices, the resulting 'matrices' \([\lambda_{(\alpha\beta)}]^\nu_\mu\) or \([\lambda_{(\alpha\beta)}]_\mu^\nu\) are fully antisymmetric. Hence \( \partial_s L = 0 \) and hence \( \Lambda = 0 \).

But there is another much simpler way to see that \( \Lambda = 0 \). In most cases, Lagrange functions are defined as scalar functions, meaning that they are by definition invariant under Lorentz transformations and rotations. For example, the Lagrange function of a free particle \( L = -mc \sqrt{-\dot{x}^\mu \dot{x}_\mu} \) is such an invariant, implying that \( \Lambda \equiv 0 \).
Knowing that $\Lambda = 0$ one can show (see exercise) that the corresponding conserved charges are given by

$$Q_{\alpha\beta} = p_\alpha x_\beta - p_\beta x_\alpha .$$

(3.82)

For spatial rotations the three charges $Q_{12}$, $Q_{13}$ and $Q_{23}$ can be interpreted as the components of the angular momentum with respect to the origin.
4. Classical Fields

4.1. Scalar fields

4.1.1. The concept of fields

A field is a set of degrees of freedom living on spacetime that is characterized by certain symmetries.

The simplest case is a classical real-valued scalar field $\phi: \mathbb{R}^{3+1} \to \mathbb{R} : x \mapsto \phi(x)$. The field is called classical if quantum effects are neglected. A classical field evolves according to a deterministic equation of motion that can be derived by means of the Lagrange formalism from a certain action. We have seen already an example of such an equation of motion, namely, the wave equation

$$\Box \phi = 0$$ (4.1)

and its massive generalization, the Klein-Gordon equation

$$\left(\Box - M^2\right)\phi = 0,$$ (4.2)

where $\Box = \partial_{\mu}\partial^{\mu}$ is the d’Alembert (quabla) operator. What would be the appropriate action that would give us this equation of motion?

Figure 4.1.: Visualization of a classical real-valued field $\phi(x)$. We can think of a field as an additional degree of freedom living on spacetime $(ct, x)$. Such a field may be plotted in a 3D representation (left) or via colorscale in a density plot (right).
Lagrange density

In the case of a point particle, the action was defined as an integral over a Lagrange function along the trajectory of the particle. Here the ‘configuration’ of the system was the path that we varied while keeping the boundary fixed. Now the ‘configuration’ of the system is the entire field, that is, the whole function $\phi(x)$ at all points in space-time, and obviously we have to integrate over the entire spacetime on which the function is defined. So the action will be of the form

$$S = \int d^4x \mathcal{L}$$

(4.3)

where $\mathcal{L}$ has the SI-unit of an action divided by a 4-volume:

$$[\mathcal{L}] = \frac{(\text{kg m}^2 \text{s}^{-1})}{(\text{m}^3 \text{s})} = \frac{\text{kg m s}^2}{\text{m}^3 \text{s}}$$

(4.4)

For this reason, the Lagrange function $\mathcal{L}$ is referred to as a Lagrange density.

In the case of a point particle the Lagrange function $\mathcal{L}$ depends on the position $x$ and the velocity $\dot{x}$ of the particle. Therefore, it is near at hand that the Lagrange density depends on the field and its first derivative. That is, $\mathcal{L}$ is expected to depend on $\phi$ and its partial derivatives $\partial_\mu \phi$. Of course there are four such partial derivatives so that we may write

$$S[\phi] = \int d^4x \mathcal{L}\left(\phi(x), \partial_0 \phi(x), \partial_1 \phi(x), \partial_2 \phi(x), \partial_3 \phi(x)\right)$$

(4.5)

or in short

$$S[\phi] = \int d^4x \mathcal{L}\left(\phi(x), \nabla \phi(x)\right),$$

(4.6)

where $\nabla$ now stands for the 4-gradient $(\partial_0, \partial_1, \partial_2, \partial_3)$. Here the argument $\phi$ in square brackets means that $S[\phi]$ is a functional. A functional is a mathematical operation which maps an entire function (namely, the field $\phi$) to a number (the corresponding action).

What about the boundary conditions? In the case of the point particle we learned that we have to specify the ending points of the trajectory and that we have to keep them fixed during variation (which plays an essential role in the step where we perform the partial integration). But where are the ending points of the field?

In classical field theory this problem is usually addressed by assuming that the field is confined to some domain $\Omega \in \mathbb{R}^{3+1}$ and that the variation of the field vanishes on the “surface” $\partial \Omega$. This domain may be finite, as shown in the figure, but it may also cover the entire spacetime, in which case the variation of the field vanishes at infinity.
4.1 Scalar fields

**Euler-Lagrange equations**

The principle of least action tells us that among all thinkable field configurations, the field configuration realized in Nature is the one for which the action is extremal (usually minimal). This means that a small variation of this configuration will not change the action to first order. In order to see what that means, let us vary the field by

\[
\phi(x) \rightarrow \phi(x) + \chi(x)
\]

(4.7)

where \(|\chi(x)| \ll |\phi(x)|\). Then we have

\[
S[\phi] \rightarrow S[\phi + \chi] = \int_\Omega d^4x \ L \left( \phi(x) + \chi(x), \nabla \phi(x) + \nabla \chi(x) \right)
\]

\[
\approx \int_\Omega d^4x \ \left[ L(\phi, \nabla \phi) + \frac{\partial L(\phi, \nabla \phi)}{\partial \phi} \chi + \frac{\partial L(\phi, \nabla \phi)}{\partial (\partial_\mu \phi)} \partial_\mu \chi \right]
\]

(4.8)

where we omitted the argument \((x)\) in the second line in all places. Hence

\[
\delta S = S[\phi + \chi] - S[\phi] \approx \int_\Omega d^4x \ \left[ \frac{\partial L(\phi, \nabla \phi)}{\partial \phi} \chi + \frac{\partial L(\phi, \nabla \phi)}{\partial (\partial_\mu \phi)} \partial_\mu \chi \right]
\]

(4.9)

Note that the second term in the integrand is actually a sum over four terms.

Again we are confronted with the problem that \(\chi\) and \(\partial_\mu \chi\) are not independent so that we cannot pull them out in front of the bracket. This problem is solved by partial integration, assuming that the variation vanishes along the boundary\(1\)

\[
\delta S \approx \int_{\partial \Omega} \frac{\partial L(\phi, \nabla \phi)}{\partial (\partial_\mu \phi)} \chi d\mu_\mu + \int_\Omega d^4x \ \left[ \frac{\partial L(\phi, \nabla \phi)}{\partial \phi} - \partial_\mu \frac{\partial L(\phi, \nabla \phi)}{\partial (\partial_\mu \phi)} \right] \chi
\]

(4.10)

Since the variation of the field \(\chi\) is arbitrary, the variation of the action is invariant only if the square bracket vanishes. This leads us to the Lagrange equations of motion, the so-called field equations

\[
\frac{\partial L(\phi, \nabla \phi)}{\partial \phi} - \partial_\mu \frac{\partial L(\phi, \nabla \phi)}{\partial (\partial_\mu \phi)} = 0.
\]

(4.11)

\(1\)This ‘partial integration’ is actually the Gaussian divergence theorem which holds in the 4D case in the same way as it does in 3D. The main point here is that the boundary contribution vanishes.
Figure 4.2.: Wave machine as a discrete model of the Klein Gordon equation. The kinetic energy of the balls would account for $-\frac{1}{2}(\partial_0 \phi)(\partial_0 \phi) = +\frac{1}{4}\phi^2$. The red horizontal springs contribute to the potential energy with $\frac{1}{2}(\nabla \phi)^2$. Finally, the green vertical springs, which drag the oscillators back to the base line, can be viewed as the mass term $-M^2\phi$ in the Klein Gordon equation. So the ‘mass’ is the Klein Gordon equation must not be confused with the mass of the oscillators.

4.1.2. Wave equation and Klein-Gordon equation

Action for the massless wave equation

What is the appropriate Lagrange density that would give us the massless wave equation $\Box \phi = 0$? Obviously this equation involves only derivatives of $\phi$, so we expect the Lagrangian to be independent of $\phi$. This means that the first term in the Lagrange equations should be zero and only the second term should give us $\Box$:

$$\Box \phi = \partial_\mu \partial^\mu \phi \propto \partial_\mu \frac{\partial \mathcal{L}(\nabla \phi)}{\partial (\partial_\mu \phi)} \Rightarrow \partial^\mu \phi \propto \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \quad (4.12)$$

The simplest solution is a quadratic expression of the form

$$\mathcal{L}(\nabla \phi) = -\frac{1}{2}(\partial_\nu \phi)(\partial^\nu \phi). \quad (4.13)$$

Thus, the wave equation is obtained from a simple quadratic Lagrange density which reminds us of the kinetic energy of a particle $\frac{m}{2}v^2$.

Action for the Klein-Gordon equation

Let us now consider the same Lagrangian with an additional potential

$$\mathcal{L}(\phi, \nabla \phi) = -\frac{1}{2}(\partial_\nu \phi)(\partial^\nu \phi) - V(\phi). \quad (4.14)$$

Written in components using the ‘mostly plus’ convention this Lagrange density reads

$$\mathcal{L}(\phi, \nabla \phi) = \frac{1}{2c^2}(\partial_t \phi)^2 - \frac{1}{2}(\nabla \phi)^2 - V(\phi), \quad \text{reminding us of } L = E_{\text{kin}} - E_{\text{pot}}, \quad \text{confirming that the signs are correct. The additional potential term turns the equations of motion into}$$

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} = -V'(\phi) + \Box \phi = 0. \quad (4.16)$$
In particular, for a quadratic potential of the form \( V(\phi) = \frac{1}{2} M^2 \phi^2 \)

\[
\mathcal{L}(\phi, \nabla \phi) = -\frac{1}{2} (\partial_\nu \phi)(\partial^\nu \phi) - \frac{1}{2} M^2 \phi^2 .
\] (4.17)

we obtain the famous Klein-Gordon equation

\[
\left( \Box - M^2 \right) \phi = 0 ,
\] (4.18)

where \( M = mc/\hbar \) is a mass parameter (see Sect. 2.2.1 on page 40):

\[
\left( \nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \frac{m^2 c^2}{\hbar^2} \right) \phi = 0 .
\] (4.19)

### 4.2. Vector fields

The scalar \( \phi(x) \) field discussed above is a single-component field. It associates with each point in space-time a certain value of the field. This values does not depend on the coordinate system and is therefore invariant under Lorentz transformations. This explains why such fields are called scalar.

Contrarily, vector fields are vector-valued functions \( \Phi(x) \) with several components \( \Phi_a(x) \). The number of the components (the dimension of the field) is in principle independent of the dimension space time. To be as general as possible, we consider a field with \( N \in \mathbb{N} \) components labeled by Latin indices \( a, b, \ldots \) running from 1 to \( N \).

What is the sense of considering a vector field with \( N \) components rather than \( N \) separate fields? The reason is that in many cases the vector field comes together with a symmetry group that mixes or rotates the components of the fields. The Lagrangian is then assumed to be invariant under these transformations. As an example we will consider the symmetry group \( O(N) \):

\( O(N) \) vector model

The simplest example would be a vector field \( \Phi(x) \in \mathbb{R}^N \) with \( N \) components \( \Phi_a(x) \) and the Klein-Gordon-like Lagrangian

\[
\mathcal{L}(\Phi, \nabla \Phi) = -\frac{1}{2} \sum_{a=1}^{N} (\partial_\nu \Phi_a)(\partial^\nu \Phi_a) - \frac{1}{2} M^2 \sum_{a=1}^{N} \Phi_a \Phi_a ,
\] (4.20)

which reduces to the scalar field Lagrangian (4.17) for \( N = 1 \). This Lagrangian has the structure of a scalar product in the index \( a \). It is therefore invariant under ordinary rotations and reflections of the \( N \) components, that is, invariant under \( O(N) \) transformations. However, this Lagrangian is trivial since it can be written as \( \mathcal{L} = \sum_{a=1}^{N} \mathcal{L}_a \), meaning that field components decouple and can be treated independently. This changes in...
interacting theories, e.g. in field theories with a quartic interaction term of the form

\[
\mathcal{L}(\Phi, \nabla \Phi) = -\frac{1}{2} \sum_{a=1}^{N} (\partial_{\mu} \Phi_{a})(\partial^{\mu} \Phi_{a}) - \frac{1}{2} M^{2} \sum_{a=1}^{N} \Phi_{a} \Phi_{a} - \frac{1}{24} \lambda \sum_{a,b=1}^{N} \Phi_{a} \Phi_{b} \Phi_{b} \Phi_{b} \tag{4.21}
\]

As can be seen, this Lagrangian is still $O(N)$-invariant in the field components, but now the components are mutually coupled via the last term. Usually such field theories are difficult to solve. However, in the limit $N \to \infty$, the situation simplifies somewhat because each field component is coupled to infinitely many other components, which can be seen as some kind of mean-field limit. This large-$N$ limit is a cornerstone in the theory of AdS/CFT correspondence (J. Erdmenger) but it is beyond the scope of the present lecture.

4.3. Noether theorem for fields

As in the case of a point particle, the Noether theorem can also be applied to fields, providing a tool that relates symmetries with the corresponding conserved quantities. The only difference is that a field has many more degrees of freedom, namely, the field values $\phi(x)$ in all points $x \in \Omega$. Therefore, the Noether theorem does not give us a finite set of conserved quantities, it rather produces an entire conserved field, the so-called Noether current, for which the conservation law is expressed in terms of a continuity equation.

**Coordinate transformations**

Let us again consider an infinitesimal coordinate transformation

\[
x \to \tilde{x} = x - \varepsilon f(x) \Rightarrow x^\mu \to \tilde{x}^\mu = x^\mu - \varepsilon f^{\mu}(x),
\]

where the displacement $\varepsilon f(x)$ may depend on $x$. As this is a passive transformation, it manipulates the coordinate system while the physical objects (the fields) remain unchanged. This is shown schematically in the adjacent figure for the simple example of a translation in one dimension. As can be seen, the changing coordinates lead to a modification of the function $\phi \to \tilde{\phi}$, and both functions are simply related by $\tilde{\phi}(\tilde{x}) = \phi(x)$. Hence we can conclude that

\[
\phi(x) = \tilde{\phi}(x - \varepsilon f(x)) \simeq \tilde{\phi}(x) - \varepsilon (\partial_{\mu} \tilde{\phi}) f^{\mu}(x)
\]

This means that an infinitesimal coordinate transformation, seen from the perspective of a fixed coordinate $x$, can always be interpreted as an infinitesimal modification of the
4.3 Noether theorem for fields

field:

\[ \tilde{\phi}(x) \simeq \phi(x) + \epsilon \Delta \phi(x). \]  

(4.24)

This transformation is called a symmetry transformation if the Lagrangian changes under this transformation by no more than a 4-divergence of a 4-vector-valued function \( \Lambda(x) \), that is:

\[ \delta L \simeq \epsilon \partial_\mu \Lambda^\mu. \]  

(4.25)

**Noether theorem for a scalar field \( \phi \)**

If the transformation (4.24) is a symmetry transformation, then the Noether theorem tells us that the so-called Noether current \( j^\mu(x) \), defined by the vector components

\[ j^\mu := \pi^\mu \Delta \phi - \Lambda^\mu \]  

(4.26)

is a conserved quantity in the sense that it obeys the continuity equation

\[ \partial_\mu j^\mu(x) = 0. \]  

(4.27)

Here \( \pi(x) \) is the conjugate field momentum given by

\[ \pi^\mu = \frac{\partial L}{\partial (\partial_\mu \phi)}. \]  

(4.28)

The field variation \( \Delta \phi(x) \) may be either given directly (e.g. for symmetries of the fields themselves) or coming from an explicit coordinate transformation \( \tilde{x} = x - \epsilon f(x) \), where \( \Delta \phi(x) \) is given by

\[ \Delta \phi(x) = (\partial_\nu \phi(x)) f^\nu(x). \]  

(4.29)

**Proof:** The proof of the relativistic Noether theorem for fields works exactly in the same way as for relativistic or non-relativistic particles. Using the abbreviation \( \delta \phi = \epsilon \Delta \phi(x) \) we first compute the change of the Lagrangian by chain rule:

\[ \delta L = \frac{\partial L}{\partial \phi} \delta \phi + \frac{\partial L}{\partial (\partial_\mu \phi)} \partial_\mu \delta \phi \]  

(4.30)

Then let us insert a zero:

\[ \delta L = \frac{\partial L}{\partial \phi} \delta \phi - \left( \frac{\partial^\mu L}{\partial (\partial^\nu \phi)} \right) \delta \phi + \left( \frac{\partial L}{\partial (\partial_\mu \phi)} \right) \partial_\mu \delta \phi \]  

(4.31)

where the first two terms give just the Euler-Lagrange equations while the third and the fourth term can be written as a common derivative by means of the product rule. Together with Eq. (4.27) we therefore arrive at

\[ \epsilon \partial_\mu \left( \frac{\partial L}{\partial (\partial^\mu \phi)} \right) \Delta \phi - \Lambda^\mu = 0 \]  

(4.32)

which completes the proof.

\[ ^2 \text{If the 4-divergence of a 4-vector-valued function is integrated over 4-volume } \Omega, \text{ we can apply Gauss theorem, giving a surface integral along } \partial \Omega. \text{ Since the variation is assumed to vanish along } \partial \Omega, \text{ this contribution does not alter the equations of motion.} \]
Noether theorem for several fields

More generally, if the Lagrangian involves several fields (for example, a complex scalar field in the form of a pair $\phi(x)$ and $\phi^*(x)$), then a given transformation will modify each field differently.

Suppose that we have a Lagrangian depending on $N$ fields $\psi_i(x)$ labeled by $i = 1 \ldots N$. Then a transformation, controlled by an infinitesimal parameter $\epsilon \ll 1$, changes each of the fields to lowest order by

$$\delta \psi_i(x) = \epsilon \Delta \psi_i(x). \tag{4.33}$$

If the transformation is a symmetry transformation (meaning that the Lagrangian is either completely invariant ($\Lambda = 0$) or that it changes $\mathcal{L}$ at most by a 4-divergence of a function $\Lambda^\mu(x)$), then the Noether theorem tells us that

$$j^\mu(x) = \sum_{i=1}^{N} \pi^\mu_{\psi_i}(x) \Delta \psi_i(x) - \Lambda^\mu(x) \tag{4.34}$$

is conserved in the sense that it obeys the continuity equation

$$\partial_\mu j^\mu(x) = 0. \tag{4.35}$$

Here $\pi_{\psi_i}(x)$ is the conjugated momentum field of $\psi_i(x)$ defined as

$$\pi^\mu_{\psi_i}(x) = \frac{\partial \mathcal{L}}{\partial (\partial_\mu \psi_i)}. \tag{4.36}$$

Noether theorem for vector fields with several components

If the Lagrangian involves a vector field with several components, as for example the electromagnetic vector potential $A^\mu$, then each component changes separately to lowest order as

$$\delta A^\mu(x) = \epsilon \Delta A^\mu(x) \tag{4.37}$$

If the transformation is a symmetry transformation, then the corresponding conserved Noether current is

$$j^\mu(x) = \pi^\mu_v(x) \Delta A^\nu(x) - \Lambda^\mu(x) \tag{4.38}$$

where

$$\pi^\mu_v(x) = \frac{\partial \mathcal{L}}{\partial (\partial_\mu A^\nu)} \tag{4.39}$$

is the momentum of the electromagnetic vector potential.
4.3.1. Energy-momentum tensor

Translation symmetry
As an example let us consider the behavior of a scalar field $\phi(x)$ under a uniform translation in space and time. In this case the displacement function $f(x)$ does not depend on $x$

$$f(x) = a = \text{const.} \quad (4.40)$$

Under a translation

$$\tilde{x} = x + \epsilon a \quad (4.41)$$

the field changes according to

$$\delta \phi(x) = \epsilon \Delta \phi(x) = \epsilon (\partial_\mu \phi(x)) a^\mu. \quad (4.42)$$

Under this change the Lagrangian $\mathcal{L}(\phi, \nabla \phi)$ varies as

$$\delta \mathcal{L} = \frac{\partial \mathcal{L}}{\partial \phi} \delta \phi + \frac{\partial \mathcal{L}}{\partial (\partial_\nu \phi)} \partial_\nu \delta \phi$$

$$= \epsilon \left( \frac{\partial \mathcal{L}}{\partial \phi} \partial_\mu \phi + \frac{\partial \mathcal{L}}{\partial (\partial_\nu \phi)} \partial_\nu \partial_\mu \phi \right) a^\mu$$

$$= \epsilon \partial_\mu \mathcal{L} a^\mu \quad (4.43)$$

so that we can identify $\Lambda = \mathcal{L} a$. Now the Noether theorem tells us that the Noether current

$$j^\mu := \pi^\mu \Delta \phi - \Lambda^\mu = \pi^\mu (\partial_\rho \phi) a^\rho - \mathcal{L} a^\mu \quad (4.44)$$

obeys the continuity equation $\partial_\mu j^\mu = 0$. This holds for all translation directions given by the vector $a$. Therefore, we expect four independent relations here if we consider translations in $x, y, z$- and in temporal direction. Choosing the direction to be the basis vector $a := e_\nu$ with the components $a^\mu = \delta^\mu_\nu$, we get a conserved Noether current of the form

$$[j_\nu]^\mu = \pi^\mu (\partial_\rho \phi) \delta_\nu^\rho - \mathcal{L} \delta_\nu^\mu = \pi^\mu \partial_\nu \phi - \delta_\nu^\mu \mathcal{L}. \quad (4.45)$$

This is a tensor with two indices

$$\mathbf{T}_\nu^\mu = \pi^\mu \partial_\nu \phi - \delta_\nu^\mu \mathcal{L} \quad (4.46)$$

which satisfies the continuity equation

$$\partial_\mu \mathbf{T}_\nu^\mu = 0.$$

Note that $\mathbf{T}$ is a proper tensor, that is, we can raise and lower the indices of $\mathbf{T}$ according to the usual rules. In the literature the tensor $\mathbf{T}$ defined above is given different names:

- Stress-energy tensor
- Stress-energy-momentum tensor
- Energy-momentum tensor

Sometimes this tensor is also denoted by the symbol $\theta$. 

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Interpretation of the energy-momentum tensor

If we want to describe the energy and the momentum of a field (a number and a 3-vector), why do we get a tensor instead of a vector in the Noether theorem? We have learned that translation symmetry in space is associated with the conservation of momentum and likewise the translation symmetry in time with the conservation of energy. In the covariant formalism we know that energy and momentum are combined in a single conserved 4-momentum $p$, but this is a vector with a single index, not a tensor with two indices. Why does the Noether theorem give us a tensor with two indices? What does this tensor mean?

The main difference is that – in contrast to a single particle – the field carries infinitely many degrees of freedom since each point $x$ in spacetime carries its own field value $\phi(x)$. In contrast to the case of a single point particle, which carries its single well-defined energy and momentum, a field is rather some kind of continuous soup that carries a different energy and momentum in each point of space-time. Therefore, a field can only be characterized by energy densities and momentum densities, and the absolute amount of energy and momentum can only be obtained by integrating over a certain volume in spacetime. In the following let us first recall how to deal with such densities:

• As a first step, let us first study a simple scalar (one–component) conserved quantity in $\mathbb{R}^3$, e.g. the mass density $\rho(\vec{x}, t)$ in a cloud of dust. In each infinitesimal volume the particles moves with a certain velocity $\vec{v} (\vec{x}, t)$. This defines a current $\vec{j}(\vec{x}, t) = \rho (\vec{x}, t) \vec{v}(\vec{x}, t)$. If we consider a certain test volume $V$, the current may lead to a loss or a gain of mass inside the volume. The total change of the mass of the particles contained inside the volume is expected to change as

$$\frac{dM}{dt} = \frac{d}{dt} \int_V d^3x \rho(\vec{x}) = - \int_{\partial V} ds \vec{j} \cdot \vec{n} \quad (4.47)$$

where $\partial V$ denotes the surface of the volume and $\vec{n}$ is the normal vector perpendicular to the two-dimensional surface element. By applying the divergence theorem we can rewrite the surface integral as a volume integral over the divergence of the current vector field, leading to

$$\frac{d}{dt} \int_V d^3x \rho(\vec{x}) = - \int_V d^3x \vec{\nabla} \cdot \vec{j}(\vec{x}). \quad (4.48)$$

Since the volume can be chosen arbitrarily, we can shrink it to point and equate the integrands, leading to the continuity equation

$$\frac{d}{dt} \rho(\vec{x}) = - \vec{\nabla} \cdot \vec{j}(\vec{x}). \quad (4.49)$$

• Next, let us put this conservation law in a relativistic language. To this end we combine the density $\rho$ and the current $\vec{j}$ in a single 4-vector $j(x)$ with the components

$$j^\mu(x) = \begin{pmatrix} \rho(\vec{x}, t) \\ \vec{j}(\vec{x}, t) \end{pmatrix}. \quad (4.50)$$
Then we can rewrite the continuity equation in the super-simple form

\[ \partial_\mu j^\mu = 0. \] (4.51)

Let us recall the interpretation of the 4-current \( j \):

- The zeroth component gives \((c \times)\) the mass density. That is, an infinitesimal volume \( dV = dx \, dy \, dz \) contains the mass \( dM = \frac{1}{c} j^0 \, dx \, dy \, dz \).

- The first component tells us how much mass passes the infinitesimal area \( dy \, dz \) in the \( yz \)-plane during an infinitesimal time \( dt \). The mass of the particles passing this area is \( dM = j^1 \, dy \, dz \, dt \).

- Similarly, the second and the third component describe the mass passing the infinitesimal areas in the \( xz \)-plane and \( xy \)-plane within time \( dt \), respectively.

The first mental abstraction required at this point is to view the mass passing the area \( dy \, dz \) within the time span \( dt \) can be considered as the mass contained in the 3-volume \( dy \, dz \, dt \).

The content of an infinitesimal 3-volume \( dA \, dt \subset \mathbb{R}^{3+1} \) can be interpreted as the flux of this content through the surface \( dA \) within the time span \( dt \).

The second mental effort required at this point is to understand the notion of a normal vector in Minkowski space. In contrast to the usual \( \mathbb{R}^3 \), where a normal vector \( d\vec{n} \) is used to describe an infinitesimal surface element \( dA \), a normal 4-vector \( d\vec{n} \) in Minkowski space describes an infinitesimal 3-volume \( dV \):

- A timelike vector \( d\vec{n} \propto e_0 \) is normal on a spatial 3-volume \( dx \, dy \, dz \).

- A spacelike vector \( d\vec{n} \propto e_1 \) is normal on a spatio-temporal 3-volume \( dy \, dz \, dt \).

- A spacelike vector \( d\vec{n} \propto e_2 \) is normal on a spatio-temporal 3-volume \( dx \, dz \, dt \).

- A spacelike vector \( d\vec{n} \propto e_3 \) is normal on a spatio-temporal 3-volume \( dx \, dy \, dt \).

Now have a clear interpretation: \( j \cdot d\vec{n} = j^\mu \, d\vec{n}_\mu \) is the infinitesimal number of particles contained in the infinitesimal (spatial or spatio-temporal) 3-volume on which \( d\vec{n} \) is normal.

Finally, let us return to the case of the energy-momentum tensor. Here we can repeat all considerations made about, the only difference being that the conserved quantity is not a scalar like the particle density \( \rho(t) \) but a vector, namely, the energy-momentum density. The first index of \( T_{\mu \nu} \) simply selects the component of this vector. For example, since the zeroth component \( p^0 = E/c \) is just the energy, the first line of the tensor \( T_{0 \mu} \) just describes the energy density and the energy flux. Likewise, the other components describe the momentum density of the moment flux, as sketched in the adjacent figure.
5. Electrodynamics

In our introductory courses on experimental physics, we were taught that all electromagnetic phenomena can be described in the framework of a unified theory proposed by Maxwell in 1864. At the core of this theory are four equations, the well-known Maxwell equations. In SI units, the Maxwell equations read

<table>
<thead>
<tr>
<th>Name</th>
<th>Integral equation</th>
<th>Differential equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gauß law for $\vec{E}$</td>
<td>$\int_{\partial V} \vec{E} \cdot d\vec{A} = \frac{1}{\epsilon_0} \int_V \rho , dV$</td>
<td>$\nabla \cdot \vec{E} = \frac{1}{\epsilon_0} \rho$</td>
</tr>
<tr>
<td>Gauß law for $\vec{B}$</td>
<td>$\int_{\partial V} \vec{B} \cdot d\vec{A} = 0$</td>
<td>$\nabla \cdot \vec{B} = 0$</td>
</tr>
<tr>
<td>Faraday’s induction law</td>
<td>$\int_{\partial A} \vec{E} \cdot d\vec{l} - \frac{d}{dt} \int_A \vec{B} \cdot d\vec{A} = 0$</td>
<td>$\nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0$</td>
</tr>
<tr>
<td>Ampère’s circuital law</td>
<td>$\int_{\partial A} \vec{B} \cdot d\vec{l} = \mu_0 \int_A \vec{j} \cdot d\vec{A}$</td>
<td>$\nabla \times \vec{B} - \mu_0 \epsilon_0 \frac{d}{dt} \int_A \vec{E} \cdot d\vec{A} + \mu_0 \epsilon_0 \frac{d}{dt} \int_A \vec{E} \cdot d\vec{A} = \mu_0 \vec{j}$</td>
</tr>
</tbody>
</table>

This looks quite complicated. But electrodynamics is in fact very simple if we understand its origin. In fact, as we will see below:

**Electrodynamics is the story of attaching a circle to all points in spacetime.**

Of course, these circles are not independent. They may be thought of as being connected, glued together, forming a geometrical continuum. As we will see below, electromagnetic fields can be interpreted as some kind of twist or internal stress of these connections.

5.1. The geometric content of electrodynamics

5.1.1. Attaching circles to each point in space-time

As we will see below, electrodynamics arises naturally from attaching a ring to each point in space-time. The rings of adjacent points in space time are glued together, forming some kind of tunnel. The idea is that twists of these tunnels along the contours indicate the presence of electric or magnetic fields.
By “attaching” we do not mean that the ring is glued to the single point of the Minkowski space in a particular place, singling out a certain position on the ring, instead the ring as a whole is associated with the reference point with the perfect symmetry along the ring. Unlike a usual dimension in space such as $x, y, z$, which extends from minus infinity to plus infinity, the circle is a closed one-dimensional manifold. The circles attached to the Minkowski space are not visible in the sense that we could experience it directly by walking along the circle just as we experience for example the motion in a particular direction in space. In other words, the attached circles are not directly perceivable, rather they may be thought of as tiny compactified degrees of freedom. Nevertheless we can feel their presence, namely, in the form of electromagnetic fields.

**Zero-dimensional space-time: The anatomy of a single ring**

To understand how electromagnetism emerges from attaching rings to space-time, let us first consider the simplest case of a zero-dimensional space-time which consists of a single point. In this case we attach a single ring to this point.

Although the circle is not glued to space-time in a particular point, we may think of ourselves as having a particular position on the circle, just in the same way as we have a particular position in space and time. This means that we are free to move on the circle. And obviously, there is a translation symmetry along the circle, just in the same way as in ordinary space, the only difference being that we return to the same point after one cycle.

In order to describe our own position on the circle, we have to introduce an appropriate coordinate system. Here it is near at hand to use an angular variable $\varphi \in [0, 2\pi]$, as sketched in Fig. 5.2. Alternatively, we could use a complex number $z \in \mathbb{C}$ on the unit circle $|z| = 1$. Both representations are related by $z = e^{i\varphi}$. In both cases we are free to choose the origin of the coordinate system at an arbitrary position place on the circle, cf. Fig. 5.2. As we will see below, this freedom to choose the origin of the coordinate system turns out to be the gauge freedom of electrodynamics.
5.1 The geometric content of electrodynamics

Figure 5.2.: Left: The ring associated with a point in space-time is a periodically closed one-dimensional space representing an intrinsic degree of freedom. Middle: A natural parameterization of the circle is to use an angle $\phi \in [0, 2\pi]$. Right: When doing so we are free to choose the origin in each circle separately.

**The circle group $U(1)$**

Translations along the ring are symmetry transformations. The corresponding group is known as the circle group or equivalently as the group of unitary transformations in one dimension $U(1)$. The circle group $U(1)$ is an Abelian (commutative) Lie group. Its group elements can be represented as follows:

**angular representation:** $\phi \rightarrow \tilde{\phi} = (\phi + \Delta \phi) \mod 2\pi$

**complex representation:** $z \rightarrow \tilde{z} = ze^{i\Delta \phi}$ (5.1)

As can be seen, the complex representation is somewhat more convenient because it does not require to perform modulo-$2\pi$ operations, it rather takes care of the periodicity by itself.

**1d space-time: Circles forming a cylinder**

Now let us assume that the supporting space is one-dimensional. Each point of this one-dimensional space is associated with an individual circle. The circles form the continuum with the topology of a cylinder:

![Figure 5.3.: Circles forming a cylinder in 1D](image)

Imagine that you were an ant living inside the cylinder. Suppose that there is no gravity. Then you will experience the perfect cylindrical symmetry and you cannot say where you are. However, there is still a clear notion of “moving straight” on this manifold\textsuperscript{\textsuperscript{1}}. The ant may either move straight in $x$-direction or in $\phi$-direction (see figure) or spiraling in a combination of both.

The one dimensional example illustrates that the attached circles

\textsuperscript{1}In differential geometry, “moving straight” means to move along a geodesic (a line without intrinsic curvature).
along a closed path in space-time form a tube. This means that space-time becomes
a network of tubes. The tubes themselves can be thought of as being elastic as if they
were made of springs. In particular, the tubes may be twisted in axial direction. The
twist are physical in nature and they are generated by the presence of electric charges.
The story of electromagnetism and electromagnetic wave is just the physics of twist
distribution in the tube network.

The adjacent figure sketches a section of straight and twisted cylinder. The main point in understanding elec-
tromagnetic phenomena is to realize that an ant living on
the surface of the cylinder, which goes straight in axial di-
rection, will follow the twist. But is there any chance for
the ant to feel or detect that twist while moving?

**Twists versus gauge**

Generally we expect the twist to change the $\varphi$-coordinate while traveling straight in
$x$-direction. However, this also depends on the choice of the local coordinate system in
each of the circles. This is demonstrated in Fig. 5.4. The upper part of the figure shows
a physically twisted cylinder together with a “straight” trajectory in axial direction that
follows the twist. If we decide to choose the origins of the local coordinate systems all
the same place, as indicated by the horizontal row of blue bullets, then the angle $\varphi(x)$
will increase by roughly $2\pi$ as we move along the trajectory. Contrarily, the lower part
of the figure shows an untwisted cylinder together with the corresponding trajectory
which now looks like a horizontal line. However, here we decided to choose a different
gauge, letting the origins of the circular coordinate systems (the blue bullets) wind
around the cylinder in the opposite sense. As a result, the angular coordinate will also
increase by roughly $2\pi$.

Note that the two situations are very different in nature: in the first case we have a physical distortion while in the second case the apparent increase of $\varphi(x)$ is solely
caused by an unreasonable but legitimate choice of the local coordinate systems. This
illustrates that $\varphi(x)$ responds not only to physical twist but also on the choice of the
gauge. In fact, we could even compensate a given twist by choosing an appropriate
counter-gauge, placing the blue dots exactly on the trajectory so that we have $\varphi(x) = 0$.
everywhere along the trajectory. This explains why an ant living inside a cylinder has no possibility to distinguish twist and gauge effects.

The whole story reminds us of active and passive transformations that we discussed in the first sections. On a one-dimensional tube the ant is not capable of distinguishing active (physical twists) and passive (choice of coordinate) transformations.

**Local gauge transformations**

In the context of electrodynamics, a gauge transformation is nothing but the change of the local coordinate systems on the $U(1)$ circles. This is illustrated in the figure on the right, where the origin of the coordinate system (the blue bullet) is moved to a different place while your physical location (yellow bullet) remains unchanged. As a result, the coordinate, namely, the angle $\phi(x)$, changes by a certain amount $\theta(x)$. More generally, in 3+1 dimensions, where $x$ is a 4-vector, a gauge transformation takes the simple form

$$\phi(x) \mapsto \bar{\phi}(x) = \phi(x) + \theta(x),$$

where the sum on the right hand side is understood modulo $2\pi$. Equivalently, we may use the complex representation defined by $z = e^{i\phi}$:

$$z(x) \mapsto \bar{z}(x) = z(x)e^{i\theta(x)}.$$  \hspace{1cm} (5.3)

**Implementing the internal circle as a fifth dimension?**

If we were to invent electrodynamics from scratch, the most natural implementation would be to consider the internal circle as a fifth dimension, in addition to the usual four dimensions of space-time. Rather than interpreting $\phi$ as some kind of field $\phi(x)$ (which is misleading), we would consider $\phi$ as a coordinate, on equal footing with $ct, x, y, z$. In such a setup, $ct, x, a, z, \phi$ would span a 5-dimensional space.

Sounds convincing? This approach as been suggested and carried out in then first half of the 20th century and is known today as Kaluza-Klein theory, a precursor of modern string theory. Remarkably, the Kaluza-Klein theory gives the full set of Maxwell equations and the correct energy-momentum tensor (called the “Kaluza miracle”).

Combining this idea with quantum theory, it was later proposed that the attached circles have a tiny circumference of about $10^{-30}$m and that the quantization of electric charge can be viewed being in correspondence with quantum-mechanical standing waves along the circle. However, this implies various problems, for example, concerning the value of the elementary charge.

Dealing with the Kaluza-Klein theory requires to fully understand the machinery of General Relativity. For this reason we will follows the standard approach and consider the electromagnetic field as something “separate” that can be described in terms of a 4-vector potential $A$. 

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The price for avoiding a fifth dimension

If we prefer to avoid the full integration of the fifth dimension, we have to pay a price. This is shown in the adjacent figure. The left panel sketches a vector space where \( \phi \) is fully integrated as a fifth dimension. Here a vector between different points (red bullets) has a perfect abstract meaning, independent of the origin of the coordinate system (blue bullet). However, if we prefer to separate the Minkowski space (horizontal) and \( \phi \) as an additional degree of freedom (vertical), it amounts to slicing the 5D space time into a bundle of 1D fibers perpendicular to a 4D space time. Now a vector between two points has no abstract meaning because the two points live in different fibers, i.e., different vector space. In each of them we could choose a different origin (blue bullet). To describe the red vector, we would need a prescription how to get from one fiber to the next. As we will see below, this prescription is provided by the electromagnetic vector potential \( A(x) \).

Connecting the fibers: The electromagnetic vector potential \( A \)

The electromagnetic potential describes the rate at which the angle \( \phi \) changes as we move "straight" in space in a certain direction. In the simplest case of a one-dimensional space discussed above, we expect that

\[
\frac{d\phi}{dx} \propto A(x) \cdot dx.
\]  

This relation can be generalized easily to higher dimensions: if we move infinitesimally in the direction \( dx \), we expect \( \phi \) to change by

\[
\frac{d\phi}{dx} \propto A(x) \cdot dx = A_\mu(x) \cdot dx^\mu.
\]  

There is still a missing proportionality factor. For reasons that we cannot fully explain at this point, this proportionality factor is just the elementary charge divided by Planck's constant:

\[
\frac{d\phi}{dx} = \frac{e}{\hbar} A(x) \cdot dx = \frac{e}{\hbar} A_\mu(x) \cdot dx^\mu.
\]  

The interpretation is the following: Suppose that we go "straight" over an infinitesimal distance in the direction \( sx \) in space-time (the yellow line in the figures above), then our coordinate \( \phi \) changes by \( d\phi \).

The vector field \( A(x) \) is just the electromagnetic vector potential that we have introduced in the last section. Now it has a clear interpretation. It tells us how to move "straight" in the network of tubes. In differential geometry, quantities like \( A(x) \) are denoted as connections because they tell us how different pieces of space (in our case the circles) are glued together and how we get from one circle to the next one.
5.1 The geometric content of electrodynamics

Gauge transformations

Instead of considering the circles as a fifth dimension of space-time, we stick here to
the usual four-dimensional space-time and consider a 4-vector field $A$ that relates the
intrinsic coordinate $\phi$ on infinitesimally neighboring circles. In many respects this is
technically simpler. But keeping the electromagnetic field as a separate entity has a
major drawback:

- The vector field $A(x)$ does have an abstract meaning with respect $SO^+(1,3)$ trans-
f ormations (Lorentz boosts and rotations), in particular $d\phi$ is invariant under co-
 ordinate transformations in the 4D space-time.

- The vector field $A(x)$ has no abstract meaning with respect to $U(1)$ trans-
 formations (gauge transformations) in the sense that $d\phi$ is not invariant under coordi-
nate transformations on the circle.

The cultural highlight of this lecture, namely, the attempt to formulate physics in ab-
stract terms (independent of the choice of the coordinates) goes to pieces here: The
4-vector potential $A(x)$ is not gauge-invariant, it rather depends on my personal choice
of the coordinates in the circles. By separating the internal degree of freedom on the
circle from the other degrees of freedom in $ct, x, y, z$, we buy simplicity, but we pay a
price, namely, a gauge dependence of the main object on which our theory is based: the
electromagnetic 4-vector potential $A(x)$.

To be more specific, let us again consider a $U(1)$ gauge transformation, namely, a
translation on the circle by a (position-dependent) difference angle $\theta$:

$$\phi(x) \mapsto \tilde{\phi}(x) = \phi(x) + \theta(x). \quad (5.7)$$

This implies that $d\phi$, the differential change of $\phi$ as we move “straight” in the direction
d$x$, is modified as follows:

$$d\phi \mapsto \tilde{d}\phi = d\phi + \nabla \theta \cdot d\mathbf{x} = d\phi + (\partial_\mu \theta) d\mathbf{x}^\mu \quad (5.8)$$

Since $d\phi = \frac{e}{\hbar} A \cdot d\mathbf{x}$ we arrive at

$$\frac{e}{\hbar} A \cdot d\mathbf{x} \mapsto \frac{e}{\hbar} \tilde{A} \cdot d\mathbf{x} = \frac{e}{\hbar} A \cdot d\mathbf{x} + \nabla \theta \cdot d\mathbf{x} \quad (5.9)$$

or in short

$$A \rightarrow \tilde{A} = A + \frac{\hbar}{e} \nabla \theta \quad (5.10)$$

In coordinates:

$$A_\mu(x) \rightarrow \tilde{A}_\mu(x) = A_\mu(x) + \frac{\hbar}{e} \partial_\mu \theta(x) \quad (5.11)$$

In most textbooks, the prefactor $\frac{\hbar}{e}$ is suppressed by setting $\theta(x) = \frac{e}{\hbar} f(x)$:

In a gauge transformation, the electromagnetic 4-vector potential $A$
changes by a gradient of a scalar function $A_\mu \rightarrow A_\mu + \partial_\mu f$.

Example: Confused? May be the following simple example is helpful:
Consider the Earth in $\mathbb{R}^3$ and a flying airplane. The position of the plane is given by a position vector $\vec{x} \in \mathbb{R}^3$ which is perfectly abstract. We can represent this vector in different coordinate systems with different components, but the vector as such is an abstract element of the vector space $\mathbb{R}^3$.

Suppose that we want to separate our description of the plane’s position into two separate parts, namely, the position on the Earth’s surface and a perpendicular “field”, namely, the height of the plane. The surface of Earth is a space $S_2$, positions on $S_2$ are perfectly abstract (and can be represented by coordinates such as longitude and latitude), but for our field, the height, we have to make a choice: Height above what? Sea level? Ground? This is the inevitable gauge freedom, a redundancy that we have introduced by separating the position into two pieces.

5.1.2. The electromagnetic field

2d space time: Detecting physical twists

As we have seen above, in a one-dimensional cylinder there is no way for an ant to distinguish between a physical twist and a gauge (see Fig. 5.4). In higher dimensions, however, the situation is totally different. The reason is that in 2D and above, the ant has much more freedom, in particular it can move on a closed contour in space-time. If we attach to each point of the closed contour a circle we get the topology of a torus (see Fig. 5.5). Therefore, moving along the close contour actually means to move on a torus.

As a remarkable fact, a closed contour allows us to discriminate between physical twists and gauge artifacts. This is illustrated in Fig. 5.6. The left panel shows an ordinary (untwisted) torus. If an ant were to move along the torus, marked by the yellow path, it would arrive exactly the same point from where it started. Contrarily, on a twisted torus it would arrive at different point, and the resulting mismatch would quantify the intensity of the twist. Note that this mismatch is physical in nature and therefore independent of the chosen gauge.

The situation reminds us of a triangle drawn on a flat piece of paper or on the sphere. While in the first case the three angles add up to $\pi$, this sum of the angles turned out to be larger than $\pi$ in the second case, reflecting the underlying curvature of the sphere.
5.1 The geometric content of electrodynamics  

Likewise, the mismatch of the angle $\varphi$ along a closed contour reflects the internal twist of the torus.

**Quantitative computation of the twist**

Let us now calculate the mismatch along a closed contour quantitatively. To this end we consider a closed path in $\mathbb{R}^2$ in the form of a square, as shown in the adjacent figure. Suppose that each side of the square has the length $2\epsilon$, extending from $-\epsilon$ to $+\epsilon$. How does the angle $\varphi$ change along this contour? In order to calculate this change, we can simply use the electromagnetic vector potential $A = (A_x, A_y)$. In fact, according to Eq. (5.5), if we move along the infinitesimal line element $dx$ of the contour, the angle $\varphi$ will just change by (setting $a = 1$)

$$d\varphi = A \cdot dx = A_{\mu} d\xi^\mu.$$ 

We can readily compute the changes along the four edges of the square:

$$\Delta \varphi_{A \to B} = \int_{-\epsilon}^{+\epsilon} A_x(x, -\epsilon) \, dx,$$

$$\Delta \varphi_{B \to C} = \int_{-\epsilon}^{+\epsilon} A_y(\epsilon, y) \, dy,$$

$$\Delta \varphi_{C \to D} = -\int_{-\epsilon}^{+\epsilon} A_x(x, \epsilon) \, dx,$$

$$\Delta \varphi_{D \to A} = -\int_{-\epsilon}^{+\epsilon} A_y(-\epsilon, y) \, dy.$$ 

The next step is to take $\epsilon \to 0$ so that $|x| \leq \epsilon$ and $|y| \leq \epsilon$. This allows us to Taylor-expand the electromagnetic potential around the origin:

$$A(x, y) = A(0, 0) + x \partial_x A(0, 0) + y \partial_y A(0, 0) + O(x^2, y^2). \quad (5.12)$$

Figure 5.6.: Left: Ordinary torus, were a straight trajectory arrives at the same point from where it started. Right: If the torus is twisted, a straight trajectory does not necessarily close.
Inserting this approximation in the integrals we obtain e.g. for the first one the expression
\[ \Delta \phi_{A \rightarrow B} \simeq \int_{-\epsilon}^{+\epsilon} \left( A_x(0,0) + x \partial_x A_x(0,0) - \epsilon \partial_y A_x(0,0) \right) \, dx. \] (5.13)

Here the second term in the integrand does not contribute because it is antisymmetric in \( x \). Similarly we can compute the other integrals, leaving us with
\[
\begin{align*}
\Delta \phi_{A \rightarrow B} &= 2\epsilon A_x(0,0) - 2\epsilon^2 \partial_y A_x(0,0) + \mathcal{O}(\epsilon^3) \\
\Delta \phi_{B \rightarrow C} &= 2\epsilon A_y(0,0) + 2\epsilon^2 \partial_x A_y(0,0) + \mathcal{O}(\epsilon^3) \\
\Delta \phi_{C \rightarrow D} &= -2\epsilon A_x(0,0) - 2\epsilon^2 \partial_y A_x(0,0) + \mathcal{O}(\epsilon^3) \\
\Delta \phi_{D \rightarrow A} &= -2\epsilon A_y(0,0) + 2\epsilon^2 \partial_x A_y(0,0) + \mathcal{O}(\epsilon^3).
\end{align*}
\] (5.14)

Altogether the resulting change is
\[ \Delta \phi = \Delta \phi_{A \rightarrow B} + \Delta \phi_{B \rightarrow C} + \Delta \phi_{C \rightarrow D} + \Delta \phi_{D \rightarrow A} = 4\epsilon^2 \left( \partial_x A_y(0,0) - \partial_y A_x(0,0) \right) + \mathcal{O}(\epsilon^3). \] (5.15)

Note that the mismatch is proportional to \( 4\epsilon^2 \), the area enclosed by the contour. At first glance it might be surprising that the mismatch scales with the area not with a circumference of the contour. To understand this have a look at Fig. 5.7. Here you see four contours arranged as \( 2 \times 2 \) tiles. Suppose that each contour detects a mismatch \( \Delta \phi \). Since the contributions along neighboring integration paths cancel each other (indicated by red color in the figure), the sum of the four integrals equals the integral along the outer contour. This means that the outer contour gives a mismatch of \( 4\Delta \phi \), hence it scales with the area and not with the circumference.

**Remark:** Alternatively, you may think of a rectangular contour of size \( \Delta x \) and \( \Delta y \). If we take \( \Delta y \to 0 \), then the contour reduces to two lines integrated in opposite direction. Clearly, these two contributions cancel one another so that the detected mismatch tends to zero. This supports that the mismatch scales with the area of the rectangle.
The geometric content of electrodynamics

5.1 The geometric content of electrodynamics

Figure 5.8.: Electrodynamics in 2+1 dimensions. In a 2+1-dimensional space there are three linearly independent orientations of closed contours. Each loop corresponds to a specific component of the electromagnetic field. In 2+1 dimensions the electric field has two components while the magnetic field has only a single component.

**Twisted loops in 3+1 dimensions: The electromagnetic field tensor**

Having understood that the detected mismatch scales with the area of the loop, it is meaningful to define the intensity of the electromagnetic field $F_{xy}$ as the detected mismatch divided by the area of the loop in the limit $\epsilon \to 0$:

$$F_{xy} := \lim_{\epsilon \to 0} \frac{\Delta \varphi}{4 \epsilon} = \partial_x A_y - \partial_y A_x. \quad (5.16)$$

So far we have considered a single loop in the two-dimensional $xy$-plane. In higher dimensional spaces such as the Minkowski space, the mismatch detection works exactly in the same way, the only difference being that we have much more freedom to choose the orientation of the detection loop. If we restrict ourselves to rectangular closed contours along the directions given by the basis vectors, say $e_\mu$ and $e_\nu$, the corresponding electromagnetic field (i.e. the intensity of the twist) is quantified by the so-called electromagnetic field tensor $F$ with the components

$$F_{\mu\nu} := \lim_{\epsilon \to 0} \frac{\Delta \varphi}{4 \epsilon^2} = \partial_\mu A_\nu - \partial_\nu A_\mu. \quad (5.17)$$

This definition is perfectly compatible with our previous definition in Eq. (3.55) on page 61, where we identified the components of the tensor with the components of the electric and magnetic fields in 3+1 dimensions as follows:

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & B_z & -B_y \\ E_y/c & -B_z & 0 & B_x \\ E_z/c & B_y & -B_x & 0 \end{pmatrix} \quad (5.18)$$

It is interesting to explore how the electromagnetic theory would look like in other dimensions. For example, if we were living in a 2+1-dimensional space-time, the same tensor would be given by

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E_x/c & -E_y/c \\ E_x/c & 0 & B \\ E_y/c & -B & 0 \end{pmatrix} \quad (5.19)$$

meaning that the electric field has two components while the magnetic field has only a single component. Each of these components corresponds to a particular orientation of
the loop, as sketched in Fig. 5.8. Having only a single $B$-component, this would imply that the magnetic force has no orientation in space.

5.1.3. Homogeneous Maxwell equations

Building a cube in which all contributions cancel

So far we have learned that the physical electromagnetic fields can be interpreted as twists in the connection of the attached circles. These twists can be detected by computing the curl (=rotation)

$$ F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu $$

(5.20)

This tensor, the so-called field tensor, can be thought of as a local infinitesimal version of a closed loop. By definition, the field tensor is antisymmetric.

It is also important to note that the closed loops are oriented, depending on their sense of circulation. If adjacent loops of the same orientation are sharing a common edge, as illustrated e.g. in Fig. 5.7, the two integrations along the common edge cancel one another.

Having identified twists along closed loops as the elementary building blocks of electrodynamics, it is interesting to note that we can always arrange six square loops as a cube in such a way that all shared edges compensate one another, as shown in the figure on the right side. Therefore, if we consider a closed 3-volume in the 3+1-dimensional Minkowski space, then the corresponding contour integrals and therewith the sum of all field values associated with the loops should vanish. This is expected to give an important conservation law which is solely of geometrical origin and unrelated to the specific physical content of electrodynamics which is defined in terms of an action.
To start with, let us consider a cube in 2+1 dimensions with its center located at \( x \), as sketched in the adjacent figure. Each edge has the length of \( 2\epsilon \), and we would like to take the limit \( \epsilon \to 0 \). For small \( \epsilon \) each face of the cube contributes to the integral with \( 4\epsilon^2 F_{\mu\nu} \) evaluated at the center of the face. For example, the front face, which extends in \( x \) and \( ct \) direction, contributes with \( 4\epsilon^2 F_{01} \), where the field tensor has to be evaluated at the center of the face \( x + \epsilon e_2 \) (and not in the center of the cube \( x \)). If \( \epsilon \) is small, we can expand

\[
4\epsilon^2 F_{01}(x + \epsilon e_2) \simeq 4\epsilon^2 (1 + \epsilon \partial_2) F_{01}(x).
\]

The same has to be done for all six faces. Suppressing the common factor \( 4\epsilon^2 \) of the face area, the six faces contribute as follows:

<table>
<thead>
<tr>
<th>Face</th>
<th>Expression</th>
<th>Approximation</th>
</tr>
</thead>
<tbody>
<tr>
<td>front</td>
<td>( F_{01}(x + \epsilon e_2) )</td>
<td>( (1 + \epsilon \partial_2) F_{01}(x) )</td>
</tr>
<tr>
<td>rear</td>
<td>( -F_{01}(x - \epsilon e_2) )</td>
<td>( (-1 + \epsilon \partial_2) F_{01}(x) )</td>
</tr>
<tr>
<td>top</td>
<td>( F_{12}(x + \epsilon e_0) )</td>
<td>( (1 + \epsilon \partial_0) F_{12}(x) )</td>
</tr>
<tr>
<td>bottom</td>
<td>( -F_{12}(x - \epsilon e_0) )</td>
<td>( (-1 + \epsilon \partial_0) F_{12}(x) )</td>
</tr>
<tr>
<td>right</td>
<td>( -F_{02}(x + \epsilon e_1) )</td>
<td>( (-1 + \epsilon \partial_1) F_{02}(x) )</td>
</tr>
<tr>
<td>left</td>
<td>( F_{02}(x - \epsilon e_1) )</td>
<td>( (1 - \epsilon \partial_1) F_{02}(x) )</td>
</tr>
</tbody>
</table>

As shown in the previous figure above, all contours along the edges compensate one another, therefore the sum of all these contributions has to vanish. As one can see, the leading order contributions on the right hand side in (5.21) cancel by themselves. The next-leading order gives the condition

\[
2\epsilon \left( \partial_0 F_{12}(x) - \partial_1 F_{02}(x) + \partial_2 F_{01}(x) \right) = 0 , \tag{5.22}
\]

This equation, which comes solely from geometrical considerations, imposes an important restriction on our physical fields.

**Parenthesis and bracket notation for index permutations**

In Special and General Relativity, many scientist love the *bracket notation*. These brackets enclose a subset of either upper or lower indices within a multiplicative term. The brackets mean that we automatically add over all permutations of the indices inside the brackets, and finally divide by the number of permutations. There are two variants:

- **Round parenthesis** where all permutations are added *symmetrically* with a positive sign
- **Square brackets** where all permutations are added *antisymmetrically* with the sign of the permutation.
Formally, if $S_n$ denotes the group of permutations of $n$ elements, this notation is defined by

$$T_{(\mu\nu\ldots)} = \frac{1}{n!} \sum_{\sigma \in S_n} T_{\sigma(\mu\nu\ldots)}$$  \hspace{1cm} (5.23)$$

and

$$T_{[\mu\nu\ldots]} = \frac{1}{n!} \sum_{\sigma \in S_n} \text{sign}(\sigma) T_{\sigma(\mu\nu\ldots)},$$  \hspace{1cm} (5.24)$$

where $n$ is the number of indices inside the bracket. For example

$$T_{(\mu\nu)} = \frac{1}{2} (T_{\mu\nu} + T_{\nu\mu})$$  \hspace{1cm} (5.25)$$

and

$$T_{[\mu\nu\rho]} = \frac{1}{6} (T_{\mu\nu\rho} - T_{\mu\rho\nu} - T_{\rho\mu\nu} + T_{\rho\nu\mu} - T_{\nu\rho\mu} - T_{\nu\mu\rho}).$$  \hspace{1cm} (5.26)$$

The same notation can be used for upper indices. The brackets can also span several tensors and they may include only some indices of higher-rank tensors, for example:

$$A^{\mu[\nu B^\rho]} = \frac{1}{2} (A^{\mu\nu} B^\rho - A^{\mu\rho} B^\nu).$$  \hspace{1cm} (5.27)$$

The parenthesis and bracket notation is frequently used to symmetrize or antisymmetrize tensorial expressions.

**Homogeneous Maxwell equation**

If we divide by $\varepsilon$, the bracket notation allows us to rewrite Eq. (5.22) in the super-compact form

$$\partial_0 F_{[12]} = 0.$$  \hspace{1cm} (5.28)$$

In 2+1 dimensions, where $F$ is a $3 \times 3$ matrix given in (5.19), this gives the following constraint:

$$\partial_0 F_{12} + \partial_1 F_{20} + \partial_2 F_{01} = 0 \implies \partial_t B + \partial_x E_y - \partial_y E_x = 0.$$  \hspace{1cm} (5.29)$$

In 3+1 dimensions, we get four equations, namely, three equations with a temporal index

$$\partial_0 F_{12} = 0 \implies \partial_t B_z + \partial_x E_y - \partial_y E_x = 0$$

$$\partial_0 F_{13} = 0 \implies -\partial_t B_y + \partial_x E_z - \partial_z E_x = 0$$

$$\partial_0 F_{23} = 0 \implies \partial_t B_x + \partial_y E_z - \partial_z E_y = 0$$  \hspace{1cm} (5.30)$$

and a single equation among the three spatial indices:

$$\partial_1 F_{23} = 0 \implies \partial_x B_x + \partial_y B_y + \partial_z B_z = 0.$$  \hspace{1cm} (5.31)$$

These equations are just the homogeneous Maxwell equations

$$\partial_t \vec{B} + \vec{\nabla} \times \vec{E} = 0, \quad \vec{\nabla} \cdot \vec{B} = 0.$$  \hspace{1cm} (5.32)$$
5.1 The geometric content of electrodynamics

To summarize, we have shown that the homogeneous Maxwell equations can be written in the form

\[ \partial_{[\mu} F_{\nu\rho]} = 0. \]  \hfill (5.33)

For proving the homogeneous Maxwell equations it is actually not necessary to consider a closed cube. In fact, it suffices to insert the definition \( F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu} = 2\partial_{[\mu} A_{\nu]} \), which is already antisymmetric. Then we see immediately that the expression vanishes because partial derivatives commute:

\[ \partial_{[\mu} F_{\nu\rho]} = \partial_{[\mu} \partial_{\nu} A_{\rho]} = 0. \]  \hfill (5.34)

As often, this mathematical derivation is much more elegant, but it lacks the intuition of “compensating twists” along a closed surface.

Let us again emphasize that we have not yet defined an action and a corresponding Lagrange density for the electromagnetic field. In deriving the field from a circle we arrived at the homogeneous Maxwell equations solely on the basis of geometry. Thus the homogeneous Maxwell equations are not “physical”, they are rather “geometrical” in nature.

The homogeneous Maxwell equations have a geometrical origin and read

\[ \partial_{[\mu} F_{\nu\rho]} = 0. \]

**The dual field tensor \( \star F \)**

Using the fully antisymmetric Levy-Civita symbols in four dimensions

\[ \epsilon_{\mu\nu\rho\tau} := \begin{cases} 1 & \text{if } (\mu\nu\rho\tau) \text{ is an even permutation of } (0123) \\ -1 & \text{if } (\mu\nu\rho\tau) \text{ is an odd permutation of } (0123) \\ 0 & \text{if } (\mu\nu\rho\tau) \text{ is not a permutation of } (0123) \end{cases} \]  \hfill (5.35)

and

\[ \epsilon^{\mu\nu\rho\tau} := -\epsilon_{\mu\nu\rho\tau} \]  \hfill (5.36)

we can define the dual field tensor

\[ \star F_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\rho\tau} F^{\rho\tau}, \quad \star F^{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\rho\tau} F_{\rho\tau} \]  \hfill (5.37)

which has the components

\[ \star F_{\mu\nu} = \begin{pmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & E_z/c & -E_y/c \\ -B_y & -E_z/c & 0 & E_x/c \\ -B_z & E_y/c & -E_x/c & 0 \end{pmatrix} \]  \hfill (5.38)
and
\[ \ast F^{\mu \nu} = \begin{pmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & E_z/c & -E_y/c \\ B_y & -E_z/c & 0 & E_x/c \\ B_z & E_y/c & -E_x/c & 0 \end{pmatrix} \] (5.39)

With the dual field tensor we can rewrite the homogeneous Maxwell equations as
\[ \partial_{\mu} F_{\nu \rho} = -\epsilon_{\mu \rho \tau} \partial_{\nu} F_{\rho \tau} = -\partial_{\nu} \ast F^{\mu \nu} = \partial_{\mu} \ast F^{\mu \nu} = 0. \] (5.40)

Consequently, the homogeneous Maxwell equations can also be written as
\[ \partial_{\mu} \ast F^{\mu \nu} = 0 \] (5.41)
and it is easy to check that they reduce again to
\[ \partial_t \vec{B} + \vec{\nabla} \times \vec{E} = 0, \quad \vec{\nabla} \cdot \vec{B} = 0. \] (5.42)

The mathematical structure, that relates \( F \) and \( \ast F \) will be discussed in the next section.

5.2. The physical content of electrodynamics

So far we have studied the geometrical content of electrodynamics. Now we are going to study the physical content of the theory. Usually, the physical content of a classical theory is defined in terms of an action and analyzed via the principle of least action.

5.2.1. The action of the electromagnetic field

*Guessing the form of the Lagrangian*

According to the heuristic principle of simplicity, the action should be given by the simplest expression that is compatible with the symmetries of the system. In the case of electrodynamics, we have two important types of symmetries, namely

- Poincaré invariance (translations, rotations, Lorentz boosts)
- Gauge invariance

As usual, the Poincaré invariance tells us that the action has to be a scalar without free indices. As for gauge invariance, recall that the physical field tensor \( F \) is gauge-invariant while the connection field \( A \) is not. This suggests that the action depends on \( F \) but not explicitly on \( A \). What is the simplest scalar that we can build from \( F \)?

The simplest scalar based on \( F \) would be the trace \( \text{Tr}(F) = F_{\mu}^{\mu} \). However, this is far too simple since
\[ F_{\mu \nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu} \] (5.43)
is anti-symmetric, implying that the trace vanishes. This suggests that we need a quadratic expression in \( F \). In fact, the simplest nontrivial quadratic scalar is a dou-
ble contraction of the form $F_{\mu\nu}F^{\mu\nu}$. And indeed, this is already what Nature decided to choose! With an appropriate pre-factor the Lagrange density of the electromagnetic field without sources in SI units reads

$$\mathcal{L}(A, \nabla A) = -\frac{1}{4\mu_0}F_{\mu\nu}F^{\mu\nu}.$$  \hfill (5.44)

By definition, this Lagrange density is Poincaré- and gauge-invariant. Note that the Lagrange density is still considered to depend on the electromagnetic vector field $A$ and its derivatives $\nabla A$ (or in components $A^\nu$ and $\partial_\mu A^\nu$).

The Lagrange density can be rewritten in terms of the electromagnetic vector field $A$ by

$$\mathcal{L}(A, \nabla A) = -\frac{1}{2\mu_0}\left((\partial_\mu A_\nu)(\partial^\mu A^\nu)-(\partial_\mu A_\nu)(\partial^\nu A^\mu)\right)$$  \hfill (5.45)

and similarly in the physical fields $\vec{E}$ and $\vec{B}$ as

$$\mathcal{L} = \frac{1}{4\mu_0}\left(\frac{2}{c^2}E^2 - 2B^2\right) = \frac{1}{2}\left(\epsilon_0E^2 - \frac{1}{\mu_0}B^2\right)$$  \hfill (5.46)

Annoyed by the factors $c, \epsilon_0, \mu_0$? This proves that you are a genuine theoretical physicist. If so, enjoy the freedom and set them to 1.

**Remark:** Note that the Lagrangian is essentially given by the difference $\frac{1}{2}(E^2 - B^2)$. This difference must not be confused with the energy density $\frac{1}{2}(E^2 + B^2)$ of electromagnetic field. This is analogous to the situation in classical mechanics, where the Lagrange function is given by $L = T - V$ while the energy is given by $H = T + V$.

**Interpretation of the action**

The Lagrangian is basically a sum over the squared twist of six contours in 01, 02, 03, 12, 13, 23-direction, as illustrated in the figure in one dimension less. The action depends on the square of the twists, just in the same way as the energy of a harmonic spring depends on the square of the elongation. In other words, the Lagrangian can be thought of as measuring the twist energy density in a given point, and thus the action is just the total energy related to the twists. By taking the square it is ensured that a positive and a negative twist accounts for the same action.

**Equations of motion without sources**

In the absence of sources (no charges or charged currents), the Lagrange density does not explicitly depend on $A$ but only on the partial derivatives of $A$. For this reason the Euler-Lagrange equations of motion simply read

$$-\partial_\rho \frac{\partial \mathcal{L}}{\partial (\partial_\rho A_\sigma)} = 0$$  \hfill (5.47)

Footnote 2: In textbooks one often finds $\mathcal{L} = \frac{1}{2}(\epsilon_0E^2 - \mu_0H^2)$, where $\hat{H} = \vec{H}/\mu_0$ is the magnetic field strength.
For the Lagrange density given in Eq. (5.45) these four equations read
\[ \frac{1}{\mu_0} \partial_\rho \left( \partial^\sigma A^\rho - \partial^\rho A^\sigma \right) = 0 \] (5.48)

or in short:
\[ \partial_\rho F^{\rho \sigma} = 0. \] (5.49)

This relation has a free index \( \sigma \), so altogether we have four relations. To interpret them recall the definition of \( F^{\mu \nu} \) in Eq. (3.60) on page 62:
\[
F^{\mu \nu} = \begin{pmatrix}
0 & E_x/c & E_y/c & E_z/c \\
-E_x/c & 0 & B_z & -B_y \\
-E_y/c & -B_z & 0 & B_x \\
-E_z/c & B_y & -B_x & 0
\end{pmatrix}.
\] (5.50)

For \( \sigma = 0 \) we get
\[ -\frac{1}{c} \left( \partial_t E_x + \partial_y E_y + \partial_z E_z \right) = 0 \Rightarrow \text{div} \vec{E} = 0 \] (5.51)

while for \( \sigma = 1, 2, 3 \) we obtain
\[
\begin{align*}
\frac{1}{c^2} \partial_t E_x - \partial_y B_z + \partial_z B_y &= 0 \\
\frac{1}{c^2} \partial_t E_y - \partial_z B_x - \partial_x B_z &= 0 \\
\frac{1}{c^2} \partial_t E_z - \partial_x B_y + \partial_y B_x &= 0
\end{align*}
\] (5.52)

where we used \( \mu_0 \epsilon_0 = 1/c^2 \). These are the two remaining Maxwell equations without source terms, derived here from the principle of least action.

The other two Maxwell equations are physical, following from the principle of least action. Without sources they are homogeneous:
\[ \mathcal{L} = -\frac{1}{4\mu_0} F_{\mu \nu} F^{\mu \nu} \Rightarrow \partial_\rho F^{\rho \sigma} = 0. \]

Electromagnetic waves in vacuum and Lorenz gauge
In the absence of sources (no currents or charges), we can rewrite Eq. (5.48) as
\[ \square A^\mu - \partial^\mu \partial_\rho A^\rho = 0. \] (5.53)

It was Maxwell’s seminal discovery that this equation admits non-trivial solutions, even without sources. These are the electromagnetic waves, and even today it is still an exciting mystery that they cover such a wide range of wavelengths (see Fig. 5.9).

Remark: When I was a child in pre-school age, of course didn't have WLAN at home, in fact, we didn’t even own a TV set, but we did have two radios, a big one with mysteriously glowing tubes and a small modern transistor radio. I was extremely puzzled that both radios
The physical content of electrodynamics

played the same music at the same time in different rooms. I asked my parents how this is possible. My parents told me that the music is carried by invisible electromagnetic waves which exist instantaneously everywhere. This left a deep impression on me.

The wave equation \((5.53)\) is valid in any gauge. In fact, it is easy to see that this equation is perfectly invariant under gauge transformations:

\[
\begin{align*}
A^\mu & \rightarrow A^\mu + \partial^\mu f \\
\Box A^\mu - \partial^\mu \partial^\rho A^\rho & \rightarrow \Box (A^\mu + \partial^\mu f) - \partial^\mu \partial^\rho (A^\rho + \partial^\rho f) \\
& = (\Box A^\mu - \partial^\mu \partial^\rho A^\rho) + (\Box \partial^\mu f - \partial^\mu \Box f) = 0
\end{align*}
\]

The gauge freedom can be exploited to choose a particular gauge in which the wave equation is easier to solve. In fact, it would be nice to get \(\Box A^\mu = 0\), because we already know how to solve this equation. To this end we have to show that we can always find a gauge in which the second term in Eq. \((5.53)\) vanishes. The led Ludvig Lorenz (not to be confused with Hendrik Lorentz, the inventor of the Lorentz transformation) to what is known as the Lorenz gauge

\[
\partial^\mu A^\mu = 0.
\]

In this gauge the wave equation reduces to

\[
\Box A^\mu = 0
\]

which is of course no longer gauge-invariant because it is only valid in one particular gauge, namely, in the Lorenz gauge.

The wave equation \(\Box A = 0\) is only valid in the Lorenz gauge \(\partial^\mu A^\mu = 0\).

\textbf{Proof}: In order to prove the Lorenz gauge, we have to show that for any given field \(A^\mu(x)\), we can find a gauge transformation that leads us to a new \(\tilde{A}^\mu(x)\) satisfying the Lorenz gauge:

\[
\tilde{A}^\mu = A^\mu + \partial^\mu f \quad \Rightarrow \quad \partial^\mu \tilde{A}^\mu = \partial^\mu A^\mu + \partial^\mu \partial^\rho f.
\]

Is it possible to find a gauge transformation (a function \(f\)) such that the right hand side of this equation vanishes? If we set the r.h.s. to zero, then we get the partial linear inhomogeneous differential equation \(\Box f = -\partial^\mu A^\mu\) that can always be solved by standard methods. This completes the proof.

\[\text{In some textbooks you also find the Gaussian gauge condition } \nabla \cdot \tilde{A} = 0. \text{ However, this gauge is only useful for special problems of electrostatics. This is the reason why we do not consider it here.}\]
Electromagnetic fields are generated by electric charges, or in other words, electric charges are the sources of the field. Here the velocity plays a crucial role: Resting charges are known to produce purely electrostatic fields while moving charges generate magnetic fields. Both aspects, resting and moving charges, can be captured in a single 4-vector field, the so-called 4-current \( j(\vec{x}) \) with the components

\[
j^\mu(\vec{x}) = \begin{pmatrix} c \rho(\vec{x}, t) \\ j^1(\vec{x}, t) \\ j^2(\vec{x}, t) \\ j^3(\vec{x}, t) \end{pmatrix}. \tag{5.57}
\]

This 4-vector field encodes the local density of the charges as well as their velocity. If \( \rho_0(\vec{x}) \) was the density of the charges in a co-moving frame (a frame in which the velocity of the charges vanishes), then the 4-current would be given by \( j = \rho_0 \vec{u} \), where \( \vec{u} \) is the 4-velocity.

Charges may be thought of as generating artificial twists of the intrinsic \( U(1) \)-rings in a given direction. For example, a resting charge produces a twist in temporal direction, as sketched in the figure. This twist in \( ct \)-direction will deform all spatio-temporal loops, for example the yellow one in \( ct \)-\( x \) direction, which may be interpreted as a static electric field in \( y \)-direction. On the other hand, a purely spatial loop, as for example the blue one in the \( xy \)-plane, will not be twisted, meaning that a resting charge does not generate a magnetic field. The situation changes, however, if the charge begins to move. Consequently, Lorentz transformations turn electric fields into magnetic ones and vice versa. This was actually the starting point for Einstein in his 1905 paper (see Fig. 5.10).

The easiest way to implement a charged source is to couple the 4-vector field \( j \) to the 4-vector field \( \vec{A} \) by simple contraction:

\[
\mathcal{L}(\vec{A}, \nabla \vec{A}) = - \frac{1}{4 \mu_0} F_{\mu\nu} F^{\mu\nu} + A_\mu j^\mu. \tag{5.58}
\]

While the geometric Maxwell equations (5.33) do not feel the action and thus remain unchanged, the physical Maxwell equations derived in Eq. (5.49) on page 98 become inhomogeneous:

\[
\partial_\mu F^{\mu\nu} = - \mu_0 j^\nu \tag{5.59}
\]

If we translate this back into the usual language we get the inhomogeneous Maxwell equations:

\[
\text{div} \vec{E} = \frac{1}{\epsilon_0} \rho_0, \quad \text{rot} \vec{B} - \mu_0 \epsilon_0 \partial_t \vec{E} = \mu_0 \vec{J} \tag{5.60}
\]
In terms of the vector field $\mathbf{A}$ in Lorenz gauge, we simply have

$$
\Box A^\mu = -\mu_0 j^\mu. \quad (5.61)
$$

**Charge conservation**

Adding the coupling term $A_\mu j^\mu$ in the Lagrangian (5.58) involves the vector potential $\mathbf{A}$. Therefore, this coupling makes only sense if the resulting equations of motion are invariant under gauge transformations. In fact, under a gauge transformation $A^\mu \to \tilde{A}^\mu = A^\mu + \partial^\mu f$ the Lagrangian changes as

$$
\mathcal{L} \to \tilde{\mathcal{L}} = \mathcal{L} + (\partial_\mu f) j^\mu. \quad (5.62)
$$

Invariance under gauge transformation means that the action $S$ remains unchanged by the extra term, i.e.

$$
\tilde{S} - S = -\int d^4x (\partial_\mu f) j^\mu = 0. \quad (5.63)
$$

Since this equation has to hold for any function $f(x)$, it seems to imply that $j^\mu = 0$ (no sources), but this is in fact too restrictive. Instead, we have to integrate by parts, assuming that the boundary terms vanish:

$$
\tilde{S} - S = \int d^4x \partial_\mu j^\mu = 0. \quad (5.64)
$$

Now we can apply the usual logic: Since we can choose any function $f(x)$, the remaining integrand has to vanish. This means that the 4-divergence of the current $j(x)$ has to vanish:

$$
\partial_\mu j^\mu = 0. \quad (5.65)
$$
This relation expresses charge conservation. This can be seen by separating the components, which just gives the continuity equation for the charge density and the charge current:

\[ \partial_t \rho(\vec{x}, t) + \nabla \cdot \vec{j}(\vec{x}, t) = 0 \]  

(5.66)

Charge conservation follows from gauge invariance.

**Energy-momentum tensor of the electromagnetic field**

Let us consider an electromagnetic field without sources. The action reads

\[ S = -\frac{1}{4} \int d^4x \, F^{\mu\nu} F_{\mu\nu} \]  

(5.67)

where \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \) is the electromagnetic field tensor. We first compute

\[ \frac{\partial F_{\mu\nu}}{\partial (\partial_\rho A_\sigma)} = \delta_\mu^\sigma \delta_\nu^\rho - \delta_\mu^\rho \delta_\nu^\sigma \]  

(5.68)

which, by raising the indices \( \mu\nu \), turns into

\[ \frac{\partial F^{\mu\nu}}{\partial (\partial_\rho A_\sigma)} = \eta^{\rho\mu} \eta^{\sigma\nu} - \eta^{\rho\nu} \eta^{\sigma\mu}. \]  

(5.69)

This allows us to compute the derivative of the Lagrange density \( \mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} \) with respect to the derivative of \( A \):

\[ \frac{\partial \mathcal{L}}{\partial (\partial_\rho A_\sigma)} = \frac{1}{4} \left( \delta_\mu^\rho \delta_\nu^\sigma - \delta_\mu^\sigma \delta_\nu^\rho \right) F^{\mu\nu} - \frac{1}{4} F_{\mu\nu} \left( \eta^{\rho\mu} \eta^{\sigma\nu} - \eta^{\rho\nu} \eta^{\sigma\mu} \right) = -F^{\rho\sigma} \]  

(5.70)

The energy-momentum tensor of the electromagnetic field

\[ T_\nu^\mu = \frac{\partial \mathcal{L}}{\partial (\partial_\mu A_\nu)} \partial_\nu A_\rho - \delta_\nu^\mu \mathcal{L} \]  

(5.71)

is therefore given by

\[ T_\nu^\mu = F^{\rho\mu} \partial_\nu A_\rho + \frac{1}{4} F_{\rho\sigma} \delta_\nu^\mu \]  

(5.72)

**5.3. Formulation of electrodynamics in differential forms**

The purpose of this optional section is to have more fun with mathematical physics. We would like to introduce advanced operations coming from the theory of differential forms, namely, the wedge product \( \wedge \), the exterior derivative \( d \), and the Hodge dual \( \star \). Using this formalism, electrodynamics can be formulated in an extremely elegant and natural way.
5.3.1. The wedge product, the exterior derivative, and the Hodge duality

From the tensor product to the wedge product

Without having introduced it explicitly, we are already familiar with the ordinary tensor product $\otimes$. For example, if $A$ and $B$ are two vectors with the components $A^\mu$ and $B^\nu$ we can form a new rank-2 tensor simply by multiplying the components:

$$C = A \otimes B \iff C_{\mu\nu} = A_\mu B_\nu \quad (5.73)$$

More generally, if $A$ and $B$ are two tensors of rank $n$ and $m$ with the components $A_{\mu_1...\mu_n}$ and $B_{\nu_1...\nu_m}$, then the tensor product $\otimes$ gives us a tensor of rank $n + m$:

$$C = A \otimes B \iff C_{\mu_1...\mu_n\nu_1...\nu_m} = A_{\mu_1...\mu_n} B_{\nu_1...\nu_m} \quad (5.74)$$

Therefore, the tensor product $\otimes$ allows us to generate tensors with more indices. This construction also works with upper and mixed indices. It should be noted that

- Tensors constructed in this way are special, that is, not all tensors $C$ can be represented as $C = A \otimes B$.
- There is no need for the tensors to be symmetric or antisymmetric.

In Physics, we often deal with oriented surface elements or oriented volume elements. These geometrical structures are intrinsically antisymmetric. For example, the well-known cross product in 3D is an antisymmetric operation. The same applies to a determinant. Therefore, mathematicians have invented a really fascinating formalism adapted to antisymmetric tensors. This formalism, which goes back to Grassmann, consists of a set of operations, called *exterior algebra*, which ensures that all tensors generated within this framework are automatically antisymmetric. In this context, one introduces a modified tensor product $\wedge$, the so-called *wedge product*, which differs from the ordinary tensor product by an additional antisymmetrization of the indices. In terms of the components, the wedge product is defined by

$$C = A \wedge B \iff C_{\mu_1...\mu_n\nu_1...\nu_m} = \frac{(n + m)!}{n! m!} A_{[\mu_1...\mu_n} B_{\nu_1...\nu_m]} \quad (5.75)$$

Note that in this expression all indices on the right hand side are anti-symmetrized inside the square bracket (for the bracket notation see Sect. 5.1.3 on page 93). For example, if we calculate the wedge product of two linear forms $A_{\mu}$ and $B_{\nu}$, we get

$$C_{\mu\nu} = (A \wedge B)_{\mu\nu} = 2 A_{[\mu} B_{\nu]} = A_\mu B_\nu - A_\nu B_\mu \quad (5.76)$$

**Remark:** Antisymmetric tensors have a direct geometrical interpretation. For example, let us consider the $\mathbb{R}^3$ with the standard basis $e_1, e_2, e_3$:

- One of these vectors alone represents a 1D object, e.g. the basis vector $e_1$ may be thought of as a line element of length 1 in $x$-direction.
- The wedge product of two vectors represents an oriented 2D object, e.g. $e_1 \wedge e_2$ is a square in the $xy$-plane with positive orientation (sense of circulation), while $e_2 \wedge e_1$ is
the same object with negative orientation. In $\mathbb{R}^3$ there are three linearly independent area elements of this kind: $e_1 \wedge e_2$, $e_1 \wedge e_3$, and $e_2 \wedge e_3$.

- The wedge product of three vectors represents an oriented 3D volume element. For example, $e_1 \wedge e_2 \wedge e_3$ is a cube. Also this cube has an orientation depending on the orientation of its faces.

- The wedge product of more than three vectors in $\mathbb{R}^3$ does not make sense, e.g. $e_1 \wedge e_2 \wedge e_3 \wedge e_4 = 0$ due to the antisymmetry. This means that the exterior algebra closes, i.e., it deals only with a finite number of objects.

In the four-dimensional Minkowski space we have more freedom, namely

- Four basis vectors $e_0, e_1, e_2, e_3$.
- Six 2D area elements $e_0 \wedge e_1$, $e_0 \wedge e_2$, $e_0 \wedge e_3$, $e_1 \wedge e_2$, $e_1 \wedge e_3$, and $e_2 \wedge e_3$.
- Four 3D volume elements $e_0 \wedge e_1 \wedge e_2$, $e_0 \wedge e_1 \wedge e_3$, $e_0 \wedge e_2 \wedge e_3$, and $e_1 \wedge e_2 \wedge e_3$.
- One 4D volume element $e_0 \wedge e_1 \wedge e_2 \wedge e_3$.

### The exterior derivative

The exterior derivative $\mathbf{d}$ is some kind of antisymmetrized derivative. Applying this derivative to an object makes only sense if the object depends on the position $\mathbf{x}$ in space (otherwise the derivative is zero). Possible objects include functions $f(\mathbf{x})$, vector fields $\mathbf{A}(\mathbf{x})$ with components $A^\mu(\mathbf{x})$ or $A_\mu(\mathbf{x})$, and antisymmetric higher-rank tensor fields $\mathbf{T}(\mathbf{x})$, for example a rank-2 tensor-field with the components $T_{\mu\nu}(\mathbf{x})$.

If we apply the exterior derivative $\mathbf{d}$ to a tensor of rank $n$, we get an tensor of rank $n + 1$, that is, we add another index which indicates the direction in which the derivative is taken. Without going into details here, we summarize here the most important special case:

- If $\mathbf{d}$ is applied to a scalar function $f(\mathbf{x})$, we get a linear form, the so-called **differential** $\mathbf{d}f(\mathbf{x}) = (\partial_\mu f(\mathbf{x})) e^\mu$. The differential has the components

  $$\left(\mathbf{d}f(\mathbf{x})\right)_\mu = \partial_\mu f(\mathbf{x}).$$

- If $\mathbf{d}$ is applied to a linear form $\mathbf{A}(\mathbf{x})$ with the components $A_{\mu\nu}$, we get a rank-2 tensor $\mathbf{dA}(\mathbf{x}) = \mathbf{d}(A^\mu(\mathbf{x}) e^\mu) = (\partial_\mu A^\mu(\mathbf{x})) e^\rho \wedge e^\sigma$ with the components

  $$\left(\mathbf{dA}(\mathbf{x})\right)_{\mu\nu} = \frac{1}{2} \left( \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) \right).$$

- If $\mathbf{d}$ is applied to an antisymmetric rank-2 tensor $\mathbf{T}(\mathbf{x})$ with the components $T_{\mu\nu}$, we get a rank-3 tensor $\mathbf{dT}(\mathbf{x}) = \mathbf{d}(T^\mu_{\mu\nu}(\mathbf{x})) e^\rho \wedge e^\sigma \wedge e^\tau$ with the components

  $$\left(\mathbf{dT}(\mathbf{x})\right)_{\mu\nu\rho} = \frac{1}{3!} \left( \partial_\mu T_{\rho\nu} - \partial_\rho T_{\mu\nu} + \partial_\rho T_{\nu\mu} - \partial_\mu T_{\rho\nu} + \partial_\nu T_{\rho\mu} - \partial_\mu T_{\rho\nu} \right).$$
An immediate consequence of the commutativity of partial derivatives $\partial_\mu \partial_\nu = \partial_\nu \partial_\mu$ is

$$d^2 = 0.$$  \hspace{1cm} (5.80)

This means that the second exterior derivative vanishes automatically. This circumstance is similar to $\nabla \times \nabla \times \ldots = 0$ in conventional vector analysis.

**Homogeneous Maxwell equations expressed in the exterior algebra**

Using the exterior algebra, we can express everything that we studies so far in a few lines:

Electrodynamics is a gauge theory of circles $U(1)$. These circles are connected, as described by a vector field $A(x)$. Physical fields (twists) are detected by a field tensor

$$F(x) = dA(x).$$ \hspace{1cm} (5.81)

As can be seen, the physical field is invariant under gauge transformations

$$A \rightarrow A + df \Rightarrow F \rightarrow F + d^2 f = 0.$$ \hspace{1cm} (5.82)

Since $F = dA$, the second derivative has to vanish, which gives the homogeneous Maxwell equations

$$dF = d^2 A = 0.$$ \hspace{1cm} (5.83)

**Hodge duality $\star$**

In $\mathbb{R}^3$ we are familiar with the concept of normal vectors representing an area. For example, if we span a parallelogram by two vectors $\vec{a}$ and $\vec{b}$, we can represent the area by a vector $\vec{c} = \vec{a} \times \vec{b}$ which is perpendicular on the area. The length of this vector quantifies the size of the area and its direction depends on the orientation (right hand rule). As long as the shape of the area does not matter, the normal vector provides a full equivalent description. In other words, there is a duality between $\vec{a}$, $\vec{b}$ and $\vec{c}$.

The Hodge duality is a formal generalization of this concept to arbitrary dimensions. Technically, the Hodge duality is a map, denoted by the symbol $\ ' \star ' $, that maps a tensor with $p$ indices to a tensor with $d - p$ indices, where $d$ is the dimension of the space.

In terms of components, the Hodge $\star$-operator is defined via the fully antisymmetric symbols $\epsilon_{\mu \nu \rho \tau}$ introduced in Eq. (5.35) on page 95. Basically, the Hodge operation contracts all indices of the given tensor with as many indices of the $\epsilon$-symbols, and the remaining free indices of the $\epsilon$-symbols are those of the resulting tensor. In the 3+1-dimensional Minkowski space, the Hodge dual works as follows:

- **Scalars $c$:**

  $$[\star c]_{\mu \nu \rho \tau} = \frac{1}{0!} \epsilon_{\tau \mu \nu \rho} c$$
• Vectors $A^\mu$:

$$[\star A]_{\mu\nu\rho} = \frac{1}{1!} \epsilon_{\tau\mu\nu\rho} A^\tau$$

• Tensors $F^{\mu\nu}$ of rank 2:

$$[\star F]_{\mu\nu} = \frac{1}{2!} \epsilon_{\tau\mu\nu\rho} F^{\rho\tau}$$

• Tensors $T^{\mu\nu\rho}$ of rank 3:

$$[\star T]_{\mu} = \frac{1}{3!} \epsilon_{\mu\nu\rho\tau} T^{\nu\rho\tau}$$

We have already seen an example of the Hodge-$\star$ duality, namely, the definition of the dual field tensor $\star F$ in Eq. (5.37) on page 95.

**Remark:** As you certainly know, in $\mathbb{R}^3$ the vector $\vec{c} = \vec{a} \times \vec{b}$ is strictly speaking not a vector but a pseudovector. This means that it transforms as a vector as long as the transformation preserves the orientation of the coordinate system but gets an additional minus sign if the orientation is reflected. Likewise, the objects produced by the Hodge-$\star$ operator are pseudotensors.

### 5.3.2. Electrodynamics expressed in differential forms

Starting point is the electromagnetic vector potential $A(x)$ which connects the $U(1)$-circles in space time. This potential is changes under gauge transformations as

$$A \to A + df \quad (5.84)$$

Because of $d^2 = 0$ the electromagnetic field tensor

$$F = dA \quad (5.85)$$

is automatically gauge-invariant. Taking another exterior derivative we get the homogeneous Maxwell equations (the geometric part):

$$dF = d^2 A = 0 \quad (5.86)$$

To derive the inhomogeneous Maxwell equations we vary the action

$$S = \int \left( -\frac{1}{2\mu_0} dA \wedge \star dA - A \wedge j \right) \quad (5.87)$$

Variation and “partial integration” yields

$$\delta S = \int \left( -\frac{1}{\mu_0} d\delta A \wedge \star dA - \delta A \wedge \star j \right)$$

$$= \int \left( +\frac{1}{\mu_0} \delta A \wedge d \star dA - \delta A \wedge \star j \right)$$

$$= \int \delta A \wedge \left( \frac{1}{\mu_0} d \star dA - \star j \right) = 0 \quad (5.88)$$
5.4 Electromagnetically charged fields

which gives the inhomogeneous Maxwell equations

\[ d \star F = \mu_0 \star j. \] (5.89)

Introducing the exterior co-derivative\(^4\) \(d^\dagger = -\star d\star\) and noting that \(\star\star = -1\) the inhomogeneous Maxwell equations may also be written as

\[ d^\dagger F = \mu_0 j. \] (5.90)

Because of \(d^\dagger^2 = 0\), we also get the continuity equation

\[ d^\dagger^2 F = 0 \Rightarrow d^\dagger j = 0, \] (5.91)

meaning that the charge current \(j\) is a conserved quantity.

Altogether we can now squeeze electrodynamics inside a single box:

\[ F = dA, \quad dF = 0, \quad d^\dagger F = \mu_0 j, \quad d^\dagger j = 0. \] (5.92)

5.4. Electromagnetically charged fields

5.4.1. From global to local gauge invariance

**Real-valued scalar field**

In the previous Section we studied the simplest relativistic classical field theory, namely, a scalar field theory with the Lagrangian (see Eq. (4.13) on page 72)

\[ L = -\frac{1}{2} \left( \partial^i \phi \right) \left( \partial^i \phi \right), \] (5.93)

In components this Lagrangian reads

\[ L = \frac{1}{2c^2} \frac{\partial \phi}{\partial t}^2 - \frac{1}{2} \left( \nabla \phi \right)^2. \] (5.94)

The first term is the kinetic energy of the field while the second term describes some kind of potential energy. In the exercises we saw that this Lagrangian can be obtained in the continuum limit of a discrete spring chain. Here the kinetic energy \(\frac{1}{2c^2} \left( \frac{\partial \phi}{\partial t} \right)^2\) corresponds to the kinetic energy of the masses in vertical

\[^4\text{More generally, } d^\dagger = \pm \star d\star \text{ with a sign depending on the dimension, signature, and the rank of the form.}\]
\( \phi \)-direction while the potential energy \( \frac{1}{2} ( \nabla \phi )^2 \) is just the potential energy of the horizontal springs. These springs can be thought of as providing a *surface tension* with the tendency to smooth the field.

Note that this Lagrangian depends only on derivatives of the field. Therefore, the absolute magnitude of the field does not matter, i.e., the system is invariant under global shifts \( \phi(x) \rightarrow \phi(x) + \text{const.} \) However, we could add additional springs that pull the field back to \( x \)-axis. In the Lagrangian, this amounts to adding a *mass term*

\[
\mathcal{L} = -\frac{1}{2} (\partial_\nu \phi) (\partial^\nu \phi) - \frac{1}{2} M^2 \phi^2
\]  

(5.95)

**Complexification: Attaching \( U(1) \)-circles to each point in space-time**

No matter whether the mass term is present or not, a real-valued field cannot account for electromagnetism, it cannot serve as a charged source, simply because there is no intrinsic ring in the form of a \( U(1) \) symmetry. In order to couple the field to electromagnetism, we have to attach the \( U(1) \) circle in each point of space-time. The easiest approach to achieve this goal is to *complexify* the field. With \( \phi(x) \in \mathbb{C} \) we can capture both aspects:

- The absolute value (radius) \(|\phi(x)|\) describes the magnitude of the field.
- The complex phase \( \text{arg}[\phi(x)] \) describes the position on the \( U(1) \)-circle.

Performing again the continuum limit (see exercise) one finds that due to Pythagoras the spring tension (the potential energy) is given by

\[
E_{\text{pot}} = \frac{1}{2} (\nabla \text{Re}[\phi])^2 + \frac{1}{2} (\nabla \text{Im}[\phi])^2
\]

The same applies to the temporal derivatives and the mass term. Hence we have basically doubled the problem by introducing two non-interacting scalar fields for the real and the imaginary part:

\[
\mathcal{L} = \frac{1}{2c^2} \left( \frac{\partial}{\partial t} \text{Re}[\phi] \right)^2 - \frac{1}{2} (\nabla \text{Re}[\phi])^2 - \frac{1}{2} M^2 \text{Re}[\phi]^2
\]

\[
+ \frac{1}{2c^2} \left( \frac{\partial}{\partial t} \text{Im}[\phi] \right)^2 - \frac{1}{2} (\nabla \text{Im}[\phi])^2 - \frac{1}{2} M^2 \text{Im}[\phi]^2
\]  

(5.96)

\(^5\)Actually it can be shown that there is no way to couple a real scalar field to the electromagnetic field in a gauge-invariant way.
Global $U(1)$ invariance

To see the invariance under global (i.e. homogeneous) $U(1)$-transformations let us rewrite the Lagrangian by

$$
\mathcal{L} = \frac{1}{c^2} \left( \frac{\partial}{\partial t} \phi^* \right) \left( \frac{\partial}{\partial t} \phi \right) - (\nabla \phi^*) (\nabla \phi) - M^2 \phi^* \phi. 
$$

(5.97)

or in covariant notation:

$$
\mathcal{L} = - (\partial_{\nu} \phi)^* (\partial^\nu \phi) - M^2 \phi^* \phi. 
$$

(5.98)

In this form it is easy to see that the Lagrangian is invariant under global phase shifts $\phi(x) \rightarrow e^{i\theta} \phi(x)$ since $\theta$ is independent of $x$ so that $e^{i\theta}$ and $e^{-i\theta}$ simply drop out.

Local $U(1)$ gauge transformations

In the beginning of this section we argued that the coordinate systems in all rings can be chosen individually. This means that we can gauge the coordinates independently in each point by $\phi(x) \rightarrow e^{i\theta(x)} \phi(x)$ with a phase shift $\theta(x)$ which depends on the position. It is an easy exercise to show that the Lagrangian [5.98] is no longer invariant under such local gauge transformations.

To restore the invariance, let us have a look at the operator in the Lagrangian which is responsible for violating gauge invariance, namely, the derivative $\partial_{\nu}$. The derivative is known to be the generator for translations. For example

$$
\exp(a \partial_x) f(x) = f(x+a)
$$

(5.99)

or in the Minkowski space

$$
\exp(a^\mu \partial_\mu) f(x) = f(x+a).
$$

(5.100)

Thus, the derivative operator $\partial_{\mu}$ generates translations in Minkowski space, or in other words, it gives us a recipe how to move straight in a given direction.

But now we have complexified the theory, having introduced an additional coordinate, namely, the complex angle $\varphi = \arg[\phi(x)]$. That is, we are no longer moving along a straight line in Minkowski space but rather along a cylinder attached to this line. If we continue to use $\partial_\mu$ as a generator for moving, then the coordinate $\varphi$ will remain constant as we move along the cylinder, for example $\varphi = 0$.

If $\varphi = 0$ is always at the bottom of the cylinder shown in the figure, then the resulting motion will be indeed straight. However, if the origin of the coordinate system for $\varphi$ is chosen irregularly along the cylinder, the resulting trajectory will be irregular.
as well, certainly not straight. Consequently, the partial derivative $\partial_\mu$ is no longer the
correct generator for straight motion, instead we have to redefine it appropriately. In
differential geometry, such a redefined derivative is called a covariant derivative.

5.4.2. Minimal coupling

**Covariant derivative**

Suppose we have chosen a gauge in which the entire field $\phi(x)$ is real, i.e. $\varphi = \text{arg}[\phi(x)] = 0$. Then we know the electromagnetic potential vanishes, hence the covariant derivative is just the ordinary one. Now let us apply a gauge transformation $\phi(x) \rightarrow e^{i\varphi(x)}\phi(x)$, and we are looking for a covariant derivative which does exactly the same as the ordinary derivative before the gauge transformation. In other words, the covariant derivative can be carried out by

- first inverting the gauge transformation that we made,
- then applying the usual derivative,
- and finally applying the gauge transformation again.

The corresponding relation reads

$$D_\mu := e^{i\varphi(x)}\partial_\mu e^{-i\varphi(x)}$$

(5.101)

and the product rule yields

$$D_\mu = \partial_\mu - i(\varphi_\mu \phi(x))$$

(5.102)

The vector potential was defined (cf. Eq. (5.11) on page 87) via

$$d\varphi(x) = e\frac{\hbar}{\hbar} A_\mu(x) \, dx^\mu,$$

(5.103)

where $e$ is the elementary charge, meaning that

$$\partial_\mu \varphi(x) = e\frac{\hbar}{\hbar} A_\mu(x).$$

(5.104)

Hence we get

$$D_\mu = \partial_\mu - ie\frac{\hbar}{\hbar} A_\mu.$$  

(5.105)

It is now very easy to couple the field $\phi(x)$ to the electromagnetic field $A^\mu$ simply by replacing the ordinary derivative with the covariant derivative.

$$\mathcal{L} = -(D_\nu \phi)^* (D^\nu \phi) - M^2 \phi^* \phi.$$  

(5.106)

As can be verified, this Lagrangian is indeed invariant under local gauge transformations (cf. exercise). Finally, we have to include the Lagrangian of the electromagnetic field itself, hence the total gauge-invariant Lagrangian reads

$$\mathcal{L} = -\frac{1}{4\mu_0} F^{\mu\nu} F_{\mu\nu} - (D_\nu \phi)^* (D^\nu \phi) - M^2 \phi^* \phi.$$  

(5.107)
Simplifying the Euler-Lagrange equations of motion

It is now straightforward to calculate the equations of motion. For the Lagrangian \( L \) which depends on the fields \( \phi, \phi^* \), and \( A^\mu \), we have actually three equations of motion, namely

\[
\begin{align*}
\frac{\partial L}{\partial \phi} - \partial_\mu \frac{\partial L}{\partial (\partial_\mu \phi)} &= 0 \\
\frac{\partial L}{\partial \phi^*} - \partial_\mu \frac{\partial L}{\partial (\partial_\mu \phi^*)} &= 0 \\
\frac{\partial L}{\partial A^\nu} - \partial_\mu \frac{\partial L}{\partial (\partial_\mu A^\nu)} &= 0 
\end{align*}
\]

(5.108)

Note that the second term

\[
(\partial_\nu \phi^*) \left(\partial_\nu^* \phi \right) = \left( \partial_\nu \phi^* + \frac{ie}{\hbar} A_\nu \phi^* \right) \left( \partial_\nu^* \phi \right) - \frac{ie}{\hbar} \left( A_\nu \phi^* (\partial_\nu \phi) - A_\nu \phi (\partial_\nu \phi^*) \right) + \frac{e^2}{\hbar^2} A^\nu A_\nu \phi^* \phi
\]

involves \( \phi, \phi^*, \partial_\nu \phi, \partial_\nu \phi^* \), and \( A^\nu \). Instead it would be much nicer to treat this term as being dependent only on \( (\partial_\nu \phi) \) and \( (\partial_\nu \phi^*) \). In fact, as we are going to prove below, the following two versions of the Lagrange equation are fully equivalent:

\[
\frac{\partial L(\phi, \phi^*, \partial_\lambda \phi, \partial_\lambda \phi^*, A_\lambda)}{\partial \phi} - \partial_\mu \frac{\partial L(\phi, \phi^*, \partial_\lambda \phi, \partial_\lambda \phi^*, A_\lambda)}{\partial (\partial_\mu \phi)} = 0
\]

(5.109)

\[
\Leftrightarrow \frac{\partial L(\phi, \phi^*, \partial_\lambda \phi, (D_\lambda \phi)^*, A_\lambda)}{\partial \phi} - D_\mu \frac{\partial L(\phi, \phi^*, \partial_\lambda \phi, (D_\lambda \phi)^*, A_\lambda)}{\partial (D_\mu \phi)} = 0
\]

(5.110)

or in short

\[
\frac{\partial L}{\partial \phi} - \partial_\mu \frac{\partial L}{\partial (\partial_\mu \phi)} = 0 \quad \Leftrightarrow \quad \frac{\partial L}{\partial \phi} - D_\mu^* \frac{\partial L}{\partial (D_\mu \phi)} = 0.
\]

(5.111)

Note that if we compute the third Lagrange equation \( \frac{\partial L}{\partial A^\nu} - \partial_\mu \frac{\partial L}{\partial (\partial_\mu A^\nu)} = 0 \), we still have to take into account that \( A^\nu \) is contained in the covariant derivative.

**Proof:** For keeping the notation compact, let us denote the usual Lagrangian written out in \( \partial_\mu \phi \) and \( \partial_\mu \phi^* \) as

\[
L = L(\phi, \phi^*, \partial_\lambda \phi, \partial_\lambda \phi^*, A_\lambda)
\]

while the Lagrangian expressed in terms of the covariant derivative \( (D_\mu \phi) \) and \( (D_\mu \phi)^* = D_\mu^* \phi^* \) is denoted as

\[
L_c = L_c(\phi, \phi^*, \partial_\lambda \phi, (D_\lambda \phi)^*, A_\lambda),
\]

where \( c \) reminds us of “covariant derivative”. Both the Lagrangians are of course identical, they only differ in the choice of the fields that consider as being independent.

For proving Eq. (5.111) we start with \( L \) in the usual form, differentiating with respect to \( \phi \). Here we have to take into account that the field \( \phi \) also occurs in the covariant derivative,
hence we apply the chain rule

\[
\frac{\partial L}{\partial \phi} = \frac{\partial L_c}{\partial \phi} + \frac{\partial L_c}{\partial (D_\mu \phi)} \frac{\partial (D_\mu \phi)}{\partial \phi} = \frac{\partial L_c}{\partial \phi} + \frac{\partial L_c}{\partial (D_\mu \phi)} \frac{\partial (\partial_\mu \phi - \frac{ie}{\hbar} A_\mu \phi)}{\partial \phi} = \frac{\partial L_c}{\partial \phi} - \frac{ie A_\mu}{\hbar} \frac{\partial L_c}{\partial (D_\mu \phi)}
\]

(5.112)

Next we differentiate with respect to \(\partial_\mu \phi\) using again the chain rule:

\[
\frac{\partial L}{\partial (\partial_\mu \phi)} = \frac{\partial L_c}{\partial (D_\nu \phi)} \frac{\partial (D_\nu \phi)}{\partial (\partial_\mu \phi)} = \frac{\partial L_c}{\partial (D_\nu \phi)} \frac{\partial (\partial_\nu \phi - \frac{ie}{\hbar} A_\nu \phi)}{\partial (\partial_\mu \phi)} = \frac{\partial L_c}{\partial (D_\nu \phi)}
\]

(5.113)

Combining both equations in the Lagrange equations of motion we get

\[
\frac{\partial L}{\partial \phi} - \partial_\mu \frac{\partial L}{\partial (\partial_\mu \phi)} = \frac{\partial L_c}{\partial \phi} - ie A_\mu \frac{\partial L_c}{\partial (D_\mu \phi)} - \partial_\mu \frac{\partial L_c}{\partial (D_\mu \phi)} = \frac{\partial L_c}{\partial \phi} - D_\mu \frac{\partial L_c}{\partial (D_\mu \phi)}
\]

(5.114)

This completes the proof. The same chain of replacements can be used to show that

\[
\frac{\partial L}{\partial \phi^*} - \partial_\mu \frac{\partial L}{\partial (\partial_\mu \phi^*)} = \frac{\partial L_c}{\partial \phi^*} - D_\mu \frac{\partial L_c}{\partial (D_\mu \phi^*)}
\]

(5.115)
5.4 Electromagnetically charged fields

5.4.3. Speculations about the Nature of the electric charge

**Planck units**

Our current description of Physics is based on the concept of continuous spaces and manifolds. For example, the Minkowski space is continuous, there is no largest and no smallest distance, no largest and no smallest time, and the space is smooth and differentiable. All this is reflected in the mathematics of vector spaces over real numbers. The same applies to masses, which are continuous between 0 and $\infty$.

The concept of the continuum implies that there is no natural reference scale, instead we have to define units. It is well known since long time that only three units are needed, namely a unit for length (meter), a unit for time (second), and a unit for mass (kilogram). These units are man-made and therefore arbitrary.

On the other hand, each theoretical breakthrough came with a new constant relating these units: Newton’s gravity brought us the gravitational constant, electrodynamics the velocity of light, and finally quantum mechanics the Planck constant:

\[
G = 6.6742 \times 10^{-11} \text{ m}^3 \text{ kg}^{-1} \text{ s}^{-2} \\
c = 2.9979 \times 10^8 \text{ m s}^{-1} \\
h = 1.0546 \times 10^{-34} \text{ kg m}^2 \text{ s}^{-1}
\]

(5.116)

It was Max Planck himself who realized that we can use these constants to define elementary units for distance, time, and mass. These units are the celebrated **Planck unit**, usually denoted by an index $P$:

Planck length: \[ l_p = \sqrt{\frac{hG}{c^3}} = 1.616 \times 10^{-35} \text{ m} \]

Planck time: \[ t_p = \sqrt{\frac{hG}{c^5}} = 5.391 \times 10^{-44} \text{ s} \]  

(5.117)

Planck mass: \[ m_p = \sqrt{\frac{hc}{G}} = 2.176 \times 10^{-8} \text{ kg} \]

The mere existence of these values is already striking. The Planck length and time are extremely tiny, and we could ignore those scales for most purposes, but there is an increasing consensus that something unusual will happen at these scales, something that we don’t know yet so that we refer to it with the vague term of **New Physics**. Roughly speaking, many physicist expect the continuum description to break down at this scale.

This is similar to a tablecloth: from a distance it looks homogeneous, but if you examine it on small scales, you will see that it is woven from threads. The distance between the threads could be something like the Planck length for our spacetime. Of course, presently this is crude speculation, but there are many problems with the concept of continuous spaces so that many physicist hope for solutions coming from a proper understanding of the Planck length.
**Elementary charge vs. Planck charge**

As for electrodynamics, the surprising fact is that there is also a Planck charge

\[
\text{Planck charge: } q_P = \sqrt{\frac{4\pi\varepsilon_0 h c}{4\pi\varepsilon_0 G}} = 1.876 \times 10^{-18} \text{ C} \tag{5.118}
\]

where \(\varepsilon_0\) is only an artifact of the SI units where we have introduced superfluous units like Ampere and Volt. Comparing this natural charge with the experimentally measured elementary charge \(e = 1.6022 \times 10^{-19} \text{ C}\) we find that the values are not so far apart. In fact, we have

\[
e = 0.0854 q_P \quad \Leftrightarrow \quad q_P = 11.71 e. \tag{5.119}
\]

Likewise, there is a Planck voltage

\[
U_P = \sqrt{\frac{e^2}{4\pi\varepsilon_0 G}} = 1.042 \times 10^{27} \text{ V} \tag{5.120}
\]

and therewith a Planck vector potential

\[
\mathbf{A}_P = \frac{U_P}{c} = \sqrt{\frac{e^2}{4\pi\varepsilon_0 G}} = \frac{\hbar}{lp q_p}. \tag{5.121}
\]

This value could be thought of as the maximal possible strength of an electromagnetic field. This leads us to the following speculative question: For such a maximal field, what is the circumference \(l\) of a loop on which the complex phase fulfills a complete cycle \(2\pi\)? This requires

\[
e \frac{\hbar}{l p} A_P l = 2\pi \quad \Rightarrow \quad e^{-\frac{\hbar}{l p} A_P l} = 1 \tag{5.122}
\]

leading to:

\[
\frac{l}{2\pi l p} = \frac{q_p}{e} \approx 11.71
\]

The right-hand side is related to the **fine structure constant**

\[
\alpha = \frac{e^2}{q_p^2} \approx 1/137 \tag{5.123}
\]

This is the only relevant parameter of electrodynamics. It quantifies how strongly the electromagnetic field interacts with elementary charged particles, here represented by the field \(\phi\).
6. Relativistic quantum mechanics

6.1. Quantum mechanics of particles without spin

Quantum mechanics is a fascinating story on its own. It is the most fundamental physical theory that we have, it is extremely accurate and nevertheless the interpretation of quantum mechanics is still controversially debated. In this course, however, we are primarily interested in the relativistic aspect of quantum theory. However, before discussing relativistic wave equations, let us go back to the Schrödinger equation and study its transformation behavior.

6.1.1. Schrödinger equation

**Definition**

Let us first recall the Schrödinger equation. The guideline for constructing this equation is the usual recipe for setting up quantum-mechanical wave equations: we start with the energy-momentum relation, e.g., for the non-relativistic free particle

\[ E = \frac{p^2}{2m}. \]  

(6.1)

Then we substitute the energy and momentum by the corresponding derivative operators:

\[ E \rightarrow i\hbar \partial_t \quad \vec{p} \rightarrow -i\hbar \vec{\nabla}. \]

(6.2)

If we let these operators act on a function \( \psi(x, t) \), this results into a differential equation:

\[ i\hbar \partial_t \psi(x, t) = -\frac{\hbar^2 \vec{\nabla}^2}{2m} \psi(x, t). \]  

(6.3)

Here the boldface symbols denote ordinary 3-vectors. This wave function is generally complex-valued, showing interesting wave-like interference phenomena. Note that it is of first order in time and second order in space. As initial condition, it therefore suffices to specify \( \psi(x, 0) \); we do not need to know the first derivative at \( t = 0 \).

The wave function \( \psi(x, t) \) is usually interpreted in such a way that \( \psi^*(x, t)\psi(x, t) \in \mathbb{R} \) is the probability density to find a particle at the point \( (x, t) \). This probabilistic interpretation requires the wave function to be normalized by

\[ \int d^d x \, \psi^*(x, t)\psi(x, t) = 1. \]

(6.4)

**Note:** Recall that \( \psi^* \psi \) is the probability density while \( \psi_t \) is a complex-valued amplitude.
Transformation of ordinary functions under Galilei transformations

In order to understand the transformation behavior of the Schrödinger equation, we first have to understand the transformation behavior of functions. Let us first consider an ordinary function, for example the temperature field $T(x, t)$ in the lecture room seen from the rest frame. How does the temperature field look like from the moving frame, i.e., what is $\tilde{T}(\tilde{x}, \tilde{t})$? Obviously, the temperature does not care about the chosen frame, so we expect that

$$\tilde{T}(\tilde{x}, \tilde{t}) = T(x, t). \quad (6.5)$$

Under a Galilei transformation (c.f. Eq. (2.7) on page 25)

$$x \rightarrow \tilde{x} = x - vt, \quad t \rightarrow \tilde{t} = t \quad (6.6)$$

we therefore get

$$\tilde{T}(\tilde{x}, \tilde{t}) = T(x + vt, \tilde{t}) \quad (6.7)$$

This is very logical: For example, if we want to measure the temperature at the origin in the moving frame, is is the same as measuring the temperature in the rest frame at the position $vt$. As usual, time is not modified in a Galilei transformation, it rather plays the role of a universal global parameter.

Boost of wave functions

Let us now turn to the Schrödinger equation. Since the Schrödinger equation is a non-relativistic wave equation, we would expect it to be invariant under Galilei transformations. More specifically, if $\psi(x, t)$ is a solution in the rest frame, we expect $\tilde{\psi}(\tilde{x}, \tilde{t})$ to be a solution of the same equation in the moving frame, i.e.,

$$-i\hbar \partial_t \psi(x, t) = -\frac{\hbar^2 \nabla^2}{2m} \psi(x, t) \Rightarrow -i\hbar \partial_t \tilde{\psi}(\tilde{x}, \tilde{t}) = -\frac{\hbar^2 \tilde{\nabla}^2}{2m} \tilde{\psi}(\tilde{x}, \tilde{t}) \quad (6.8)$$

According to the example of the temperature field discussed above, we expect the transformed wave function to be given by $\tilde{\psi}(\tilde{x}, \tilde{t}) = \psi(x, t)$. However, the story is not that simple. Since the derivative operators transform under Galilei transformations (cf. Eq. (2.11) on page 27) according to

$$\tilde{\partial}_t = \partial_t + v \cdot \nabla, \quad \tilde{\nabla} = \nabla \quad (6.9)$$

it is easy to see that the Schrödinger equation is not form-invariant by itself: while $\nabla^2$ remains invariant, the first derivative picks up an extra term. Therefore, assuming that $\tilde{\psi}(\tilde{x}, \tilde{t}) = \psi(x, t)$ we find that

$$-i\hbar (\partial_t + v \cdot \nabla) \psi(x, t) \neq -\frac{\hbar^2 \nabla^2}{2m} \psi(x, t) \quad (6.10)$$

So does the Schrödinger equation violate Galilei invariance?

The solution is the following: Something is wrong with our naive comparison of the wave function with a temperature field. This can already be seen from a physical point of view: A resting particle has momentum zero, hence the function $\psi(x, t)$ has no
waviness in space. From the viewpoint of the moving frame, however, the particle is moving, hence the the wave function should exhibit oscillations in space. It is clear that the naive transformation law \( \tilde{\psi}(\tilde{x}, \tilde{t}) = \psi(x, t) \) cannot hold in this case.

On the other hand, the wave function tells us that the probability density of finding a particle is given by \( \rho(x, t) = \psi^* \psi(x, t) \), and it is clear that this density should behave like the temperature field in the previous example, i.e., we must have \( \tilde{\rho}(\tilde{x}, \tilde{t}) = \rho(x, t) \).

This implies that \( \psi(x, t) \) and \( \tilde{\psi}(\tilde{x}, \tilde{t}) \) can differ at most by a complex phase which cancels out in the probability density. In fact, it turns out (see exercise) that we have to boost by the following prescription:

\[
\tilde{\psi}(\tilde{x}, \tilde{t}) = \exp\left(-\frac{imv \cdot x}{\hbar} + \frac{imv^2 t}{2\hbar}\right) \psi(x, t) = \exp\left(-\frac{imv \cdot \tilde{x}}{\hbar} - \frac{imv^2 \tilde{t}}{2\hbar}\right) \psi(\tilde{x} + v \tilde{t}, \tilde{t})
\]

(6.11)

As one can verify directly, this boosted wave function does obey the transformed wave equation

\[-i\hbar \partial_t \tilde{\psi}(\tilde{x}, \tilde{t}) = -\frac{\hbar^2 \nabla^2}{2m} \psi(\tilde{x}, \tilde{t}) .\]

To summarize, we have found that the Schrödinger equation is not form-invariant under Galilei transformations but requires to boost the wave function by suitable (position- and time-dependent) complex phases.

### 6.1.2. Klein-Gordon equation

Already in the early days of quantum mechanics, people tried to find relativistic pen- dants. Historically the first step was to simply take the relativistic energy-momentum relation and to apply the aforementioned replacement rules. As we will see below, this leads us to the Klein-Gordon equation.

**Definition**

In the relativistic setting the replacement rule for energy and momentum reads:

\[
p \rightarrow -i\hbar \partial_t \quad \Leftrightarrow \quad p^\mu = -i\hbar \partial_\mu .
\]

(6.12)

In order to write this down in components, recall that

\[
p^\mu = \begin{pmatrix} E/c \\ p_x \\ p_y \\ p_z \end{pmatrix}, \quad p_\mu = \begin{pmatrix} -E/c^T \\ p_x \\ p_y \\ p_z \end{pmatrix}, \quad \partial_\mu = \begin{pmatrix} \frac{1}{c} \partial_t \\ \partial_x \\ \partial_y \\ \partial_z \end{pmatrix}
\]

(6.13)

so that we recover the ordinary replacement rules

\[
E \rightarrow i\hbar \partial_t, \quad \vec{p} \rightarrow -i\hbar \vec{\nabla} .
\]

(6.14)

\[\text{In the ‘mostly minus’ convention we would have } p^\mu = +i\hbar \partial_\mu .\]
If we apply \( (6.12) \) to the relativistic energy-momentum relation
\[
p^\mu p_\mu = -m^2 c^2 \tag{6.15}
\]
and let it act on a scalar field \( \phi(x) \), we get the massive Klein-Gordon equation
\[
-\hbar^2 \partial^\mu \partial_\mu \phi(x) = -m^2 c^2 \phi(x) \quad \Rightarrow \quad (\Box - M^2)\phi(x) = 0, \tag{6.16}
\]
where \( M = \frac{mc}{\hbar} \) is the mass parameter.

**The problem of potentially negative probability currents**

Initially the Klein-Gordon equation was dismissed since it turned out that in contrast to the Schrödinger equation its probability density is no longer positive definite. This can be shown as follows: In order to obtain the probability density, we multiply the wave equation by \( \phi^*(x) \) from the left, and likewise we multiply the adjoint wave equation \( (\Box - M^2)\phi^*(x) = 0 \) by \( \phi(x) \) from the left. Since each expression vanishes, their difference will also vanish:
\[
\phi^*(\Box - M^2)\phi - \phi(\Box - M^2)\phi^* = 0. \tag{6.17}
\]
In this difference the mass terms drop out, hence
\[
\phi^* \partial^\mu \partial_\mu \phi - \phi \partial_\mu \partial^\mu \phi^* = 0. \tag{6.18}
\]
Since
\[
\partial_\mu (\phi^* \partial^\mu \phi) = (\partial_\mu \phi^*)(\partial^\mu \phi) + \phi^*(\partial_\mu \partial^\mu \phi)
\]
\[
\partial_\mu (\phi \partial^\mu \phi^*) = (\partial_\mu \phi)(\partial^\mu \phi^*) + \phi(\partial_\mu \partial^\mu \phi^*)
\]
we can rewrite (6.18) as the gradient of a current:
\[
\partial_\mu (\phi^* \partial^\mu \phi - \phi \partial^\mu \phi^*) = \partial_\mu j^\mu = 0. \tag{6.20}
\]
This is nothing but a relativistic continuity equation of the form
\[
-\frac{1}{c^2} \partial_t (\rho \phi^* \phi - \rho \phi \phi^* \phi) + \nabla \cdot (\rho \phi^* \nabla \phi - \rho \phi \nabla \phi^*) = 0 \tag{6.21}
\]
or in short
\[
\partial_t \rho + \nabla \cdot \vec{j} = 0. \tag{6.22}
\]
As one can see, the probability density is no longer positive definite. In fact, the Klein-Gordon equation admits solutions with positive and with negative energy, corresponding to matter and antimatter. At the time of the discovery, the existence of antimatter was not yet known and this is the reason why the Klein-Gordon equation was initially dismissed. Later it was realized that it can be used to describe bosons.

Not only the potentially negative probability density posed a serious problem, but
also the second derivative and time. Compared to the Schrödinger equation, the second derivative in time requires a completely different interpretation of a quantum wave function because the state of a particle is determined not only by \( \phi(\vec{x}, t) \) but also by the first derivative \( \partial_t \phi(\vec{x}, t) \). In order to return to the usual interpretation of quantum-mechanical wave functions, it would be desirable to formulate the wave equation as a set of first-order partial differential equations.

6.2. Dirac equation

6.2.1. Motivation of the Dirac equation

*Warm-up exercise: Dirac equation in 0+1 dimensions*

As a warm-up exercise let us start from the Klein-Gordon equation in 0+1 dimensions. Here we have no spatial degrees of freedom, meaning that there is only a second temporal derivative:

\[
\left( \frac{1}{c^2} \partial_t^2 + M^2 \right) \phi(t) = 0. \tag{6.23}
\]

This second-order differential equation can be converted into a system of two first-order differential equations by defining \( \chi(t) = \frac{i}{M} \partial_t \phi(t) \) and writing

\[
\begin{align*}
    i \frac{1}{c} \partial_t \chi(t) - M \phi(t) &= 0 \\
    i \frac{1}{c} \partial_t \phi(t) - M \chi(t) &= 0.
\end{align*} \tag{6.24}
\]

As can be seen easily, the Klein-Gordon equation can be retrieved by inserting second equation into the first one. Defining the vector

\[
\psi(t) = \begin{pmatrix} \phi(t) \\ \chi(t) \end{pmatrix} \tag{6.25}
\]

we can write this system of equations in the form

\[
\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \phi(t) \\ \chi(t) \end{pmatrix} - M \begin{pmatrix} \phi(t) \\ \chi(t) \end{pmatrix} = 0 \tag{6.26}
\]

or in short

\[
(i \gamma_\nu \partial_t - M) \psi(t) = 0. \tag{6.27}
\]

We can also choose the opposite way and use Eq. (6.27) as an ansatz: Iterating the equation given above two times one obtains the relation

\[
M^2 \psi(t) = -\frac{1}{c^2} \gamma^2 \partial_t^2 \psi(t). \tag{6.28}
\]
If we want each component of the vector $\psi(t)$ to separately obey the Klein-Gordon equation \((\frac{1}{c^2} \partial_t^2 + M^2)\psi = 0\) (see Eq. (6.23)) we are led to the condition
\[
\gamma^2 = 1.
\] (6.29)
This is so to say the ‘algebra’ that the matrix $\gamma$ has to obey. There are different possible representations. The simplest one is the one-dimensional representation $\gamma = \pm 1$. Then we simply get $(\pm i \partial_t - M)\psi(t) = 0$ with the solution $\psi \propto e^{\pm iM t}$.

Moreover, there are various two-dimensional representations. For example, we could set $\gamma$ to one of the three Pauli matrices
\[
\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\] (6.30)
since $(\sigma^x)^2 = (\sigma^y)^2 = (\sigma^z)^2 = 1$. Each of these representations results into a different representation of the wave function. For example, $\gamma = \sigma^x$ would give us the representation in Eq. (6.26).

**Dirac equation in 3+1 dimensions**

In contrast to the example discussed above, the gradient $\partial^\mu$ in Minkowski space has four components and has to be implemented covariantly, i.e., we have to contracted with another 4-vector. Therefore, the simplest generalization of Eq. (6.27) would be a system of equations of the form
\[
(i\gamma^\mu \partial_\mu - M)\psi(x) = 0.
\] (6.31)
Here $\gamma^0, \gamma^1, \gamma^2, \gamma^3$ are four operators acting in an auxiliary $k$-dimensional space, called spinor space. In this space, they are represented as $k \times k$ matrices and likewise $\psi(x)$ is a function with $k$ components. The contraction with the $\gamma$-matrices is so important in theoretical physics that a special notation has been introduced, the so-called Feynman slash notation or Feynman dagger
\[
\not{\partial} = \gamma^\mu \partial_\mu = \gamma_\mu \partial^\mu.
\] (6.32)
Of course, the slash can be applied not only to the gradient but to any 4-vector. So we may for example write
\[
\not{p} = \gamma^\mu p_\mu = \gamma_\mu p^\mu.
\] (6.33)
Using this notation the Dirac equation reads
\[
(i\not{\partial} - M)\psi = 0.
\] (6.34)
In order to determine the $\gamma$-matrices we iterate this equation two times, just as we did in the previous example:
\[
M^2\psi(x) = -\not{\partial}\not{\partial}\psi(x) = -\gamma^\mu \gamma^i \partial_\mu \partial_i \psi(x).
\] (6.35)
Because of the given dispersion relation, we expect each component of the wave function to obey the Klein-Gordon equation $(\Box - M^2)\psi = 0$, hence the right side of this equation has to be equal to $\Box\psi$:

$$-\gamma^\mu \gamma^\nu \partial_\mu \partial_\nu \psi(x) = \Box \psi(x) = \eta^{\mu\nu} \partial_\mu \partial_\nu \psi(x) \quad (6.36)$$

Comparing both sides it seems at first glance that $\gamma^\mu \gamma^\nu = -\eta^{\mu\nu} \mathbb{1}_k$, where $k$ is the dimension of the auxiliary space of the gamma matrices. However, this conclusion is premature. The reason is that the commutativity of the partial derivatives allows us to symmetrize the expression on the left side, leading to the weaker condition

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = -2\eta^{\mu\nu} \mathbb{1}_k, \quad (6.37)$$

Therefore, the algebra for the $\gamma$-matrices is given by the anti-commutator

$$\{\gamma^\mu, \gamma^\nu\} = -2\eta^{\mu\nu} \mathbb{1}_k. \quad (6.38)$$

Please be aware that there are two spaces at work: The $1 + 3$-dimensional Minkowski space (here in coordinate indices $\mu, \nu$) and the $k$-dimensional auxiliary space, later denoted as spinor space.

**Remark:** The minus sign on the right hand side of (6.38) is caused by the ‘mostly plus’ convention used throughout this course.

### 6.2.2. Representation of the $\gamma$-matrices

**Dimension of the Dirac algebra**

In the following we would like to determine suitable matrix representations for the $\gamma$-matrices. As a preliminary exercise, let us consider Pauli matrices. Pauli matrices obey a very similar algebra, namely, the *Pauli algebra*

$$\{\sigma^i, \sigma^j\} = 2\delta^{ij} \mathbb{1}_k. \quad (i, j = x, y) \quad (6.39)$$

**Remark:** As we mentioned earlier, an algebra is a set of ‘letters’ (operators, matrices) that can be used to form ‘words’ (monomials). The ‘words’ span a linear space where each word can be thought of as a separate dimension. The elements of this ‘word space’ are linear combinations of ‘words’ (polynomials). The ‘word space’ is restricted by a set of rules (algebraic relations), which are often given in the form of certain commutation relations.

The Pauli algebra is build on to basic ‘letters’, namely, the operators $\sigma^x$ and $\sigma^y$. These ‘letters’ can be used to form ‘words’, the monomials of the algebra. Not all monomials are linearly independent since the commutation relations (6.39) establish linear dependencies among the words. In the case of the Pauli algebra, we have in total four
monomials which can be classified as follows:

<table>
<thead>
<tr>
<th>Category</th>
<th>Monomial</th>
<th>Number</th>
</tr>
</thead>
<tbody>
<tr>
<td>Scalar S:</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Vector V:</td>
<td>$\sigma^x, \sigma^y$</td>
<td>2</td>
</tr>
<tr>
<td>Pseudoscalar P:</td>
<td>$\sigma^x \sigma^y = i\sigma^z$</td>
<td>1</td>
</tr>
<tr>
<td>total number:</td>
<td></td>
<td>$M=4$</td>
</tr>
</tbody>
</table>

The number of words, i.e., the dimension of the linear space spanned by the words, provides a lower boundary for the minimal matrix dimension $k \times k$ of a representation that captures the whole structure of the algebra, namely $k^2 \geq M$ since we need at least as many matrix elements as linearly independent monomials. In the case of the Pauli algebra, where we get four monomials, we therefore need at least $2 \times 2$ matrices, and indeed, the Pauli matrices are $2 \times 2$ matrices.

In the case of the Dirac algebra, the situation is similar. Here we have 16 linearly independent monomials, namely all products of $\gamma$-matrices where the indices increase (since the commutation relations allows them to be interchanged):

<table>
<thead>
<tr>
<th>Category</th>
<th>Monomial</th>
<th>Number</th>
</tr>
</thead>
<tbody>
<tr>
<td>Scalar S:</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Vector V:</td>
<td>$\gamma^0, \gamma^1, \gamma^2, \gamma^3$</td>
<td>4</td>
</tr>
<tr>
<td>Tensor T:</td>
<td>$\gamma^0\gamma^1, \gamma^0\gamma^2, \gamma^0\gamma^3, \gamma^1\gamma^2, \gamma^1\gamma^3, \gamma^2\gamma^3$</td>
<td>6</td>
</tr>
<tr>
<td>Pseudovector A:</td>
<td>$\gamma^0\gamma^1\gamma^2, \gamma^0\gamma^1\gamma^3, \gamma^0\gamma^2\gamma^3, \gamma^1\gamma^2\gamma^3$</td>
<td>4</td>
</tr>
<tr>
<td>Pseudoscalar P:</td>
<td>$\gamma^0\gamma^1\gamma^2\gamma^3$</td>
<td>1</td>
</tr>
<tr>
<td>total number:</td>
<td></td>
<td>$M=16$</td>
</tr>
</tbody>
</table>

Therefore, we need at least $4 \times 4$ matrices. And in fact, the three most important representations are defined in terms of $4 \times 4$ matrices, as we will see in the below.

**The $\gamma^5$ matrix**

In addition, it is useful to define the product of the four gamma matrices. This product plays a role analogous to $\sigma^z$ for the Pauli matrices and is defined as

$$
\gamma^5 := i\gamma^0\gamma^1\gamma^2\gamma^3.
$$

(6.40)

**Note:** Why not $\gamma^4$? This comes from old times when the coordinates were still enumerated from 1...4 instead of 0...3. Physicists are sort of conservative, so they kept the index 5.

As we will see in more detail below, this matrix plays an important role in the theory of fermions when it comes to *chirality*. At this point let us only summarize the most important mathematical properties:
• \( \gamma^5 \) is Hermitean, i.e., \( (\gamma^5)^\dagger = \gamma^5 \).
• \( (\gamma^5)^2 = \gamma^5 \gamma^5 = 1 \), therefore the eigenvalues of \( \gamma^5 \) are \( \pm 1 \).
• \( \gamma^5 \) anticommutes with all other \( \gamma \)-matrices: \( \{ \gamma^5, \gamma^\mu \} = \gamma^5 \gamma^\mu + \gamma^\mu \gamma^5 = 0 \).

**Dirac representation – the standard choice**

The so-called Dirac representation is defined by

\[
\begin{align*}
\gamma^0 &= \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, & \gamma^j &= \begin{pmatrix} -\sigma^j & \sigma^j \end{pmatrix}, & \gamma^5 &= \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix},
\end{align*}
\] (6.41)

where \( 1 \) is a 2 \( \times \) 2 identity matrix and \( \sigma^j \) denote the usual 2 \( \times \) 2 Pauli matrices

\[
\sigma^1 = \sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\] (6.42)

Explicitly the Dirac representation matrices are given by:

\[
\begin{align*}
\gamma^0 &= \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, & \gamma^j &= \begin{pmatrix} -\sigma^j & \sigma^j \end{pmatrix}, & \gamma^5 &= \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix},
\end{align*}
\] (6.43)

**Proof:** In order to prove the validity of the Dirac representation, we could simply insert the matrices into the commutation relations and check them by hand. Another elegant approach is to use tensor products. For example, the tensor product of two 2 \( \times \) 2 matrices is defined by

\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} ae & af & be & bf \\ ag & ah & bg & bh \\ ce & cf & de & df \\ cg & ch & dg & dh \end{pmatrix}.
\] (6.44)

It is important to realize that the tensor product commutes with the ordinary matrix multiplication:

\[
(A \otimes B)(C \otimes D) = (AC) \otimes (BD)
\] (6.45)

Using these rules the \( \gamma \)-matrices in the Dirac representation can expressed as

\[
\begin{align*}
\gamma^0 &= \sigma^x \otimes 1_2, & \gamma^j &= i \sigma^y \otimes \sigma^j \quad (j = 1, 2, 3).
\end{align*}
\] (6.46)

This enables us to prove the Dirac algebra by using the commutation relations of the Pauli matrices.
matrices:

1. \( \{\gamma^0, \gamma^0\} = 2(\gamma^0)^2 = 2(\sigma^2) \otimes 1 = 2 \)
2. \( \{\gamma^0, \gamma^j\} = (\sigma_i^0 \sigma^j) \otimes \sigma^i + (i \sigma^0 \sigma^j) \otimes \sigma^i \), \(\sigma^0 \otimes \sigma^j = 0\)  \(6.47\)
3. \( \{\gamma^j, \gamma^k\} = (i \sigma^j) \otimes (\sigma^k) = 2 \delta_{jk} \)

Note that this proof is independent of the specific representation of the Pauli matrices.

Weyl representation – for massless fermions

The Weyl representation, also known as the chiral representation differs from the Dirac representation only in the definition of \(\gamma^0\):

\[
\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma^j = \begin{pmatrix} 0 & -\sigma^j \\ \sigma^j & 0 \end{pmatrix}, \quad \gamma^5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{6.48}
\]

One can show that it differs from the standard Dirac representation only by a unitary transformation. As we will see below, the Weyl representation is particularly useful for the description of massless fermions.

Majorana representation – for real-valued spinors

The Majorana representation is a special representation in which all matrices are purely imaginary. This means that the Dirac equation \((i \gamma^\mu \partial_\mu - M) \psi(x) = 0\), where we have an \(i\) in front of \(\gamma^\mu\), turns into a purely real system of partial differential equations.

\[
\gamma^0 = \begin{pmatrix} \sigma^2 & \sigma^2 \\ \sigma^2 & \sigma^2 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} i \sigma^3 & i \sigma^3 \\ -i \sigma^3 & -i \sigma^3 \end{pmatrix}, \quad \gamma^2 = \begin{pmatrix} \sigma^2 & -\sigma^2 \\ \sigma^2 & -\sigma^2 \end{pmatrix}, \quad \gamma^3 = \begin{pmatrix} \sigma^2 & -\sigma^2 \\ \sigma^2 & -\sigma^2 \end{pmatrix}, \quad \gamma^5 = \begin{pmatrix} -\sigma^2 & \sigma^2 \\ -\sigma^2 & \sigma^2 \end{pmatrix} \tag{6.49}
\]

In the literature this representation is not unique but you will find several variants. They all have in common that the \(\gamma\)-matrices are purely imaginary. Therefore, when inserted into the Dirac equation \((i \gamma^\mu \partial_\mu - M) \psi = 0\), they are multiplied by \(i\) so that we get a linear system of first-order differential equations with real coefficients.

This highlights an important fact: Coming from the Schrödinger equation we started out from the assumption that the wave function \(\psi(x)\) should complex-valued. However, this is not really necessary: The Majorana representation proves that we could in principle work with a completely real-valued 4-component wave function. However, later, when coupling the Dirac fermion to an electromagnetic field, we would have to complexify the wave function anyway in order to add the internal \(U(1)\)-circle in each component of the wave function at any point in space-time.

Remember: In the Dirac equation, we could in principle have a completely real-valued wave function. This requires to use the so-called Majorana representation of the \(\gamma\)-matrices. From the physical point of view the use of complex wave function is only required when coupling the fermion to an electromagnetic field.
Ettore Majorana was certainly one of the most bizarre personalities in Theoretical Physics. Who else could abandon Physics from one day to the other, disappearing from this world at the age of 31 without leaving a trace? If you like you may read the article by Barry. L. Holstein, J. Phys Conf. Ser 173 (2009) 012019

6.2.3. Spinors

**Spinor space**

All the representations introduced above are four-dimensional. Note that this 4D space is an auxiliary internal space with the purpose to provide a realization of the anticommutation relations (6.38). We refer to this space as the *spinor space*. Roughly speaking, a spinor is a vector living in spinor space, and $\psi(x)$ in the Dirac equation is just a spinor field. It is important to realize that the spinor space, although being four-dimensional, differs from the Minkowski space.

**Internal spinor space $\neq$ Minkowski space**

**Form-invariance of the Dirac equation under Lorentz transformation**

On the other hand, it is easy to see that both spaces are closely related. In fact, we expect that the Dirac equation $(i\gamma^\mu \partial_\mu - M)\psi(x) = 0$ should be form-invariant under Lorentz transformations

$$x^\mu \rightarrow \tilde{x}^\mu = \Lambda^\mu_\nu x^\nu, \quad \partial_\mu \rightarrow \tilde{\partial}_\mu = \Lambda^\mu_\nu \partial_\nu$$

and obviously this requires the Dirac $\gamma$-matrices to transform like a 4-vector

$$\gamma^\mu \rightarrow \tilde{\gamma}^\mu = \Lambda^\mu_\nu \gamma^\nu.$$  \hspace{1cm} (6.51)

Clearly, this mixing of $\gamma$-matrices makes only sense if the new $\tilde{\gamma}$-matrices obey the same commutation relations as the old ones, or in other words, if they are again a representation of the Dirac algebra (6.38). This can be verified as follows:

$$\{\tilde{\gamma}^\mu, \tilde{\gamma}^\nu\} = \{\Lambda^\mu_\rho \gamma^\rho, \Lambda^\nu_\sigma \gamma^\sigma\} = \Lambda^\mu_\rho \Lambda^\nu_\sigma \{\gamma^\rho, \gamma^\sigma\} = 2\eta^\mu\sigma.$$  \hspace{1cm} (6.52)

This confirms that the new $\tilde{\gamma}$-matrices are in fact a valid representation of the Dirac algebra. However, it is clear that these new matrices differ from the old ones ($\tilde{\gamma}^\mu \neq \gamma^\mu$), meaning that the Dirac equation would not be form-invariant under Lorentz transformations. Therefore, in order to restore form invariance under Lorentz transformations, we have to transform the internal spinor space as well, as we will see in the following paragraph.
Covariant spinor transformation

Let us again formulate the problem: If \( \psi(x) \) is a solution of the Dirac equation

\[
(i\gamma^\mu \partial_\mu - M) \psi = 0 \tag{6.53}
\]

and if we define \( \tilde{x}^\mu = \Lambda^\mu_\nu x^\nu \) and \( \tilde{\gamma}^\mu = \Lambda^\mu_\nu \gamma^\nu \), then it is clear that \( \psi(x) \) is also solution of

\[
(i\tilde{\gamma}^\mu \tilde{\partial}_\mu - M) \tilde{\psi} = 0. \tag{6.54}
\]

But this is not what we want because these differential equations differ from the previous ones as they involve different \( \tilde{\gamma} \) matrices. What we would prefer instead is to find a solution \( \tilde{\psi} \) for the same set of equations (having the original \( \gamma \)-matrices without the tilde in place) expressed in the new coordinates \( \tilde{x} \):

\[
(i\tilde{\gamma}^\mu \tilde{\partial}_\mu - M) \tilde{\psi} = 0. \tag{6.55}
\]

The idea is that this solution \( \tilde{\psi}(\tilde{x}) \) differs from the original solution by some transformation \( S \) in spinor space:

\[
\tilde{\psi} = S\psi. \tag{6.56}
\]

Inserting this ansatz into Eq. (6.55) we get

\[
(i\gamma^\mu \partial_\mu - M) S\psi = (i\gamma^\mu \Lambda^\mu_\nu \partial_\nu - M) S\psi = 0. \tag{6.57}
\]

This equation may be multiplied from the left with the inverse spinor transformation \( S^{-1} \), giving

\[
S^{-1}(i\gamma^\mu \Lambda^\mu_\nu \partial_\nu - M) S\psi = 0 \tag{6.58}
\]

\[
\Rightarrow (iS^{-1}\gamma^\mu \Lambda^\mu_\nu \partial_\nu - M) \psi = 0. \tag{6.59}
\]

On the other hand we know that

\[
(i\gamma^\nu \partial_\nu - M) \psi = 0
\]

and therefore it would the sufficient for Eq. (6.57) to hold if

\[
\gamma^\nu = S^{-1}\gamma^\mu \Lambda^\mu_\nu S. \tag{6.60}
\]

If we multiply the whole equation with \( \Lambda^\rho_\nu \) from the left and use the orthogonality of the Lorentz transformation \( \Lambda^\rho_\nu \Lambda^\rho_\mu = \delta^\rho_\mu, \) we can bring the Lorentz transformation to the left side

\[
\Lambda^\rho_\nu \gamma^\nu = S^{-1}\gamma^\rho S. \tag{6.61}
\]

This relation determines the spinor transformation \( S \) for a given Lorentz transformation \( \Lambda \).

Calculation of the spinor transformation matrix \( S \)

For given \( \Lambda \), solving Eq. (6.61) for \( S \) is very difficult because \( S \) enters quadratically and also involves a matrix inversion. But there is a simple way out: we know that Lorentz transformations form a Lie group, meaning that we can expand them around the iden-
6.2 Dirac equation

Dirac equation

tical transformation, and the same is expected to hold for the transformation $S$. Thus the idea is to study Eq. (6.61) in the first-order approximation and then to extrapolate to the general result by taking the exponential.

Let us start with an infinitesimal Lorentz transformation of the form

$$\Lambda^\mu_\nu = \delta^\mu_\nu + e^{(\alpha\beta)} \lambda_{(\alpha\beta)}^\mu_\nu + O(e^2),$$  \hfill (6.62)

where $\lambda_{(\alpha\beta)}$ are the six generators of rotations and Lorentz transformation (see Eq. (2.90) on page 44) and $e^{(\alpha\beta)}$ denote the corresponding infinitesimal ‘angles’. Similarly, we assume that the spinor transformation can be expressed in the infinitesimal form

$$S = 1 + \frac{i}{2} e^{(\alpha\beta)} \sigma_{(\alpha\beta)} + O(e^2),$$  \hfill (6.63)

where the $\sigma_{(\alpha\beta)}$ are six yet unknown generators in spinor space and where we inserted a factor $i/2$ just for later convenience. Fortunately, the inversion of the matrix $S$ is trivial in the first-order approximation, all what we have to do is to flip the sign of the linear term:

$$S^{-1} = 1 - \frac{i}{2} e^{(\alpha\beta)} \sigma_{(\alpha\beta)} + O(e^2).$$  \hfill (6.64)

Now we insert Eqs. (6.62)-(6.64) into Eq. (6.61) and compare the first-order terms on both sides:

$$\gamma^\mu + e^{(\alpha\beta)} \lambda_{(\alpha\beta)}^\mu_\nu \gamma^\nu + O(e^2) = \left(1 - \frac{i}{2} e^{(\alpha\beta)} \sigma_{(\alpha\beta)} + O(e^2)\right) \gamma^\mu \left(1 + \frac{i}{2} e^{(\alpha\beta)} \sigma_{(\alpha\beta)} + O(e^2)\right)$$

$$\Rightarrow e^{(\alpha\beta)} \lambda_{(\alpha\beta)}^\mu_\nu \gamma^\nu = \left[\gamma^\mu, \frac{i}{2} e^{(\alpha\beta)} \sigma_{(\alpha\beta)}\right]$$  \hfill (6.65)

Since this correspondence has to hold for all ‘angles’ $e^{(\alpha\beta)}$, we can conclude that

$$\lambda_{(\alpha\beta)}^\mu_\nu \gamma^\nu = \frac{i}{2} \left[\gamma^\mu, \sigma_{(\alpha\beta)}\right].$$  \hfill (6.66)

This is now a linear equation in the generators $\sigma_{(\alpha\beta)}$ which could be solved by standard methods of linear algebra. But here we take a simpler somewhat more intuitive approach. First we insert the definition of the generators $\lambda_{(\alpha\beta)}$ in Eq. (2.90) on page 44:

$$\gamma^\mu \sigma_{(\alpha\beta)} - \sigma_{(\alpha\beta)} \gamma^\mu = -2i \left(\delta^\mu_\alpha \eta_{\beta\nu} - \delta^\mu_\beta \eta_{\alpha\nu}\right) \gamma^\nu$$  \hfill (6.67)

$$\Rightarrow \sigma_{(\alpha\beta)} = \frac{i}{2} [\gamma_\alpha, \gamma_\beta] = \begin{cases} i\gamma_\alpha \gamma_\beta & \text{if } \alpha \neq \beta \\ 0 & \text{otherwise} \end{cases}$$  \hfill (6.68)

Proof: As one can see directly, the solution is correct for $\alpha = \beta$. For $\alpha \neq \beta$ we can verify this result by inserting (6.68) into (6.67), that is, we have to show that

$$i \left(\gamma^\mu \gamma_\alpha \gamma_\beta - \gamma_\alpha \gamma_\beta \gamma^\mu\right) = -2i \left(\delta^\mu_\alpha \eta_{\beta\nu} - \delta^\mu_\beta \eta_{\alpha\nu}\right).$$

Here we have to use the anticommutation relations $\{\gamma^\mu, \gamma^\nu\} = -2\eta^{\mu\nu}$. If $\mu \neq \alpha$ and $\mu \neq \beta$ we have $\{\gamma_\alpha, \gamma^\mu\} = \{\gamma_\beta, \gamma^\mu\} = 0$, hence the l.h.s. vanishes, and so does the right side of
If $\mu = \alpha$ we can show that
\[
\gamma^\alpha \gamma_\alpha \gamma_\beta - \gamma_\alpha \gamma^\alpha \gamma_\beta = \{\gamma^\alpha, \gamma_\alpha\} \gamma_\beta = -2 \gamma_\beta.
\]

The proof for $\mu = \beta$ is analogous.

So far we have established a relation between the generators $\lambda_{(\alpha\beta)}$ for rotations and Lorentz boosts and the corresponding generators $\sigma_{(\alpha\beta)}$ for the accompanying spinor transformation that would be necessary to restore form invariance of the Dirac equation.

\textbf{Remark:} What we have done is a very common trick in Theoretical Physics. We have a non-linear relationship between transformations that we cannot solve directly. However, we can study the infinitesimal case and consider the equation to first order in $\epsilon$. This gives a linear relation between the corresponding generators.

To get the relation between the (non-infinitesimal) full transformations, all what is left to do is to exponentiate the generators. That is, a rotation or Lorentz boost (see Eq. (2.93) on page 45)
\[
\Lambda = \exp \left( \sum_{0 \leq \alpha < \beta \leq 3} \theta_{(\alpha\beta)} \lambda_{(\alpha\beta)} \right) \tag{6.69}
\]
has to be accompanied by the spinor transformation
\[
\mathbf{S} = \exp \left( \frac{i}{2} \sum_{0 \leq \alpha < \beta \leq 3} \theta_{(\alpha\beta)} \sigma_{(\alpha\beta)} \right) \tag{6.70}
\]

\textbf{Spinor rotation}

Let us, for example, consider a rotation in the $xy$-plane by the angle $\theta = \theta_{(12)}$. Here we get
\[
\Lambda_{(12)}(\theta) = \begin{pmatrix} 1 & \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{6.71}
\]
and
\[
\mathbf{S}_{(12)}(\theta) = \begin{pmatrix} e^{i\theta/2} & 0 & 0 \\ 0 & e^{-i\theta/2} & 0 \\ 0 & 0 & e^{i\theta/2} \end{pmatrix} \tag{6.72}
\]

Surprisingly, for a rotation by $2\pi = 360^\circ$ we get $\Lambda = 1$ but $S = -1$. This is a hallmark of spin $1/2$: A full rotation results into a phase factor $-1$. Here is a list how particles with different spin behave under spatial rotations:

- spin 0 (Scalar) | fully invariant under rotations
- spin $\frac{1}{2}$ (Fermion) | back after $4\pi = 720^\circ$
- spin 1 (Photon) | back after $2\pi = 360^\circ$
- spin 2 (Graviton) | back after $\pi = 180^\circ$
6.2.4. Transformation behavior of scalar fields and spinor fields

How a Lorentz transformation acts on a scalar field

Let us now investigate how the Lorentz transformation combined with a spinor transformation acts on a function, for example on a scalar field \( \phi(x) \). Recall the simple case of translation: if we move the coordinates to the left, we have to move the argument of the function to the right and vice versa (cf. page 17):

\[
x \to x - a \\
\Rightarrow \phi(x) \to \phi(x + a)
\]

(6.73)

In components the same relation can be expressed in terms of an exponential function acting on the function

\[
\phi(x^\mu e_\mu) \to \phi((x^\mu + a^\mu)e_\mu) = e^{a^\mu \partial_\mu} \phi(x^\mu e_\mu),
\]

(6.74)

that is, \( e^{a^\mu \partial_\mu} \) is the operator which translates the function by the distance \( a \).

Similarly, in the case of a Lorentz boost or a rotation we have

\[
x \to \Lambda^T x \\
\Rightarrow \phi(x) \to \phi(\Lambda x),
\]

(6.75)

where \( \Lambda^T = \Lambda^{-1} \). In components we can write

\[
\phi(x^\mu e_\mu) \to \phi(\Lambda^\nu x^\nu e_\mu).
\]

(6.76)

Is it possible to express this operation as an exponential as in the case of translation? To address this question let us consider the case of an infinitesimal transformation

\[
\Lambda = 1 + \epsilon^{(\alpha\beta)} \lambda_{(\alpha\beta)},
\]

(6.77)

where \( \lambda_{(\alpha\beta)} \) are the generators of \( SO^+(3,1) \) for rotations / Lorentz boosts in the \( (\alpha\beta) \)-plane while \( \epsilon^{(\alpha\beta)} \) are the corresponding infinitesimal parameters. Written components the above equation reads

\[
\phi(x^\mu e_\mu) \to \phi(x^\mu e_\mu + \epsilon^{(\alpha\beta)} \lambda_{(\alpha\beta)} \mu x^\nu e_\mu),
\]

(6.78)

where the second term in the argument on the right hand side can be understood as a sum of infinitesimal displacements. We can therefore pull them in front of the function by using a derivative:

\[
\phi(x^\mu e_\mu) \to \left(1 + \epsilon^{(\alpha\beta)} \lambda_{(\alpha\beta)} \mu x^\nu \partial_\nu \right) \phi(x^\mu e_\mu),
\]

(6.79)

This can be exponentiated quite easily, giving

\[
\phi(x) \to \exp \left( \theta^{(\alpha\beta)} \lambda_{(\alpha\beta)} \mu x^\nu \partial_\nu \right) \phi(x).
\]

(6.80)
Example: Rotations and the angular momentum

For example, a rotation by the angle $\theta = \theta(12)$ in the $xy$-plane reads

$$\phi(x) \to \exp \left( \theta \left( -x \partial_y + y \partial_x \right) \right) \phi(x). \quad (6.81)$$

This can also be written as

$$\phi(x) \to \exp \left( -i \frac{\theta}{\hbar} \left[ \vec{r} \times \vec{p} \right] \right) \phi(x). \quad (6.82)$$

where

$$L_z = \vec{r} \times \vec{p} = -i\hbar (x \partial_y - y \partial_x) \quad (6.83)$$

is the $z$-component of the quantum-mechanical orbital angular momentum operator.

How a Lorentz transformation acts on a spinor field

If the scalar field $\phi(x)$ is replaced by a 4-component spinor field $\psi(x)$, then the transformation of the internal degrees of freedom has to be included as well:

$$\psi(x) \to \exp \left( -i \frac{\theta}{\hbar} \left[ \vec{r} \times \vec{p} \right] \right) \psi(x). \quad (6.84)$$

Now, for a rotation in the $xy$-plane, we get two contributions, namely, the orbital angular momentum operator

$$L_z = i\hbar \lambda(12) \mu^\nu \partial_\mu x^\nu \quad (6.85)$$

whose eigenvalues are integer multiples of $\hbar$, as well as the spin operator

$$S_z = -\frac{\hbar}{2} \tau(12) \quad (6.86)$$

with eigenvalues $\pm \hbar/2$. The two operators commute so that we can add them without any problems. Their sum is called the total angular momentum

$$J_z := L_z + S_z \quad (6.87)$$

and similarly the $x$ and $y$ component, which together constitute a vector operator

$$\vec{J} = \vec{L} + \vec{S}. \quad (6.88)$$

Proof: In order to calculate the eigenvalues of $L_z$ and $S_z$, let us again consider a rotation in the $(12)=xy$-plane. What are the eigenvalues of the angular momentum operator $L_z$? To answer this question let us go to cylindrical (or polar) coordinates $x = \cos \phi$ and $y = \sin \phi$. In these coordinates the angular momentum operator is given by

$$\vec{L} = -i\hbar \left[ \vec{r} \times \vec{p} \right] = -i\hbar \partial_\phi. \quad (6.88)$$

The eigenfunctions of this operator are the $2\pi$-periodic waves $e^{im\phi}$ with $m \in \mathbb{Z}$, hence the eigenvalues $hm$ are integer multiples of $h$. 

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What are the eigenvalues of the spin operator \( -\frac{\hbar^2}{2} \gamma_{(12)} \)? Recall that for \( \alpha \neq \beta \) we have

\[
\gamma_{(\alpha \beta)} = \frac{i}{2} [\gamma_\alpha, \gamma_\beta] = i \gamma_\alpha \gamma_\beta
\]

\[
(\gamma_{(12)})^2 = \gamma_1 \gamma_2 \gamma_1 \gamma_2 = + \gamma_1 \gamma_1 \gamma_2 \gamma_2 = 1.
\]

Consequently the matrix \( \gamma_{(12)} \) must have the eigenvalues \( \pm 1 \) and therefore the whole term contributes with a half-integer spin \( \pm \hbar/2 \).

In the Dirac equation, orbital angular momenta are quantized as integer multiples of \( \hbar \), while the intrinsic spin is \( \pm \hbar/2 \).

### 6.2.5. Plane-wave solutions and their interpretation

**Plane-wave solution of the Dirac equation**

Let us now determine the plane-wave solution of a free particle with momentum \( p \). We start with a simple plane-wave ansatz of the form\(^2\)

\[
\psi(x) = u(p) e^{i \bar{\hbar} \varepsilon p \cdot x}
\]

i.e., \( \psi(x) = u(p) e^{i \bar{\hbar} \varepsilon p \cdot x} \), where \( \varepsilon \) is a dimensionless constant to be determined later. If we insert this ansatz into the Dirac equation

\[
(i \gamma^\mu \partial_\mu - M) \psi = 0
\]

and using that \( M = mc/\hbar \) we get the relation

\[
\left( -\frac{1}{\hbar} e^{i \varepsilon \bar{\hbar} p \cdot x} \right) u(p) = 0 \quad \Rightarrow \quad -e \gamma^\mu p_\mu u(p) = mc u(p)
\]

which looks like an eigenvalue problem. Let us assume that the \( \gamma \) matrices are given in the Dirac standard representation and that the components of the 4-momentum are given by \( p_\mu = (-E/c, p_x, p_y, p_z) \) as usual. Introducing the notation

\[
p^\pm = p_x \pm ip_y
\]

the matrix \( -\gamma^\mu p_\mu \) turns out to be given by

\[
-\gamma^\mu p_\mu = \begin{pmatrix}
E/c & 0 & -p_z & -p_-
0 & E/c & -p_+ & p_z
p_z & p_- & -E/c & 0
p_+ & -p_z & 0 & -E/c
\end{pmatrix}.
\]

This matrix has four eigenvalues

\[
\pm \sqrt{E^2/c^2 - p_+ p_- - p_z^2} = \pm \sqrt{E^2/c^2 - p^2}
\]

\(^2\)The positive sign in the exponential \( e^{i \bar{\hbar} \varepsilon p \cdot x} \) comes from the ‘mostly plus’ convention.
Relativistic quantum mechanics

(each two-fold degenerate), compatible with the relativistic energy-momentum relation
\[ E^2 = p^2c^2 + m^2c^4. \]

This implies that
\[
\begin{align*}
\epsilon_1 = \epsilon_2 &= +1 \quad \text{(solutions with positive energy)} \\
\epsilon_3 = \epsilon_4 &= -1 \quad \text{(solutions with negative energy)}
\end{align*}
\] (6.93)

The corresponding (non-normalized) Eigenvectors for a particle with positive energy \( +E \) read
\[
\begin{align*}
u_1(p) &= \begin{pmatrix}
\frac{E}{c} + mc \\
0 \\
p_z \\
p_+
\end{pmatrix}, &
u_2(p) &= \begin{pmatrix}
0 \\
\frac{E}{c} + mc \\
p_- \\
-p_z
\end{pmatrix},
\end{align*}
\] (6.94)

while the eigenvectors for particles with negative energy \( -E \) are given by
\[
\begin{align*}
u_3(p) &= \begin{pmatrix}
p_z \\
p_+ \\
\frac{E}{c} + mc \\
0
\end{pmatrix}, &
u_4(p) &= \begin{pmatrix}
p_- \\
-p_z \\
0 \\
\frac{E}{c} + mc
\end{pmatrix},
\end{align*}
\] (6.95)

**Velocity and momentum**

Let us now have a closer look at the solutions obtained above. The solution with positive energy \( (\epsilon = +1) \) are of the usual form
\[
\psi(x) = u(p)e^{i\bar{p} \cdot \vec{x}}.
\] (6.96)

Let us for simplicity consider a wave in \( x \)-direction, i.e., \( p_y = p_z = 0 \). In which direction does the wave move? To answer this question, we may look for the motion of a “wave crest”, or those points of the wave where the complex phase is an integer multiple of \( 2\pi \).

To this end we have to solve
\[
0 = p_\mu x^\mu = -\frac{E}{c}ct + p_x x \quad \Rightarrow \quad x = \frac{E}{p_x}t.
\] (6.97)

Here we read off that the velocity of the wave \( v = E/p_x \) is positive. Next we want to determine the physical momentum of the wave. In quantum mechanics, the momentum operator is given by
\[
P = -i\hbar \nabla
\] (6.98)

and obviously
\[
P\psi = -i\hbar \nabla e^{i\bar{p}_\mu x^\mu} = -i\hbar \nabla e^{i(-Et + \bar{p} \cdot \vec{x})} = +\bar{p}\psi,
\] (6.99)

meaning that the velocity and the momentum of the wave point in the same direction. However, turning to the solution with negative energy, things become weird. On the one hand, by solving \( 0 = p_\mu x^\mu \) we still have a positive velocity. On the other hand, the

---

3These eigenvalues are expected to be independent of the chosen representation of the \( \gamma \)-matrices.
physical momentum
\[
P \psi = -i\hbar \nabla e^{-i p \cdot x / \hbar} = -i\hbar \nabla e^{-i (Et + \vec{p} \cdot \vec{x})} = -\vec{p} \psi
\]

(6.100)
is now negative. This would imply that particles with negative energy travel to the right, carrying momentum to the left. This would have dramatic consequences. For example, particles with negative energy in a container would generate a negative pressure, since when a particle is reflected at the wall, the wall is dragged in the direction from where the particle came.

**Remember:** Relativistic particles with negative energy have the paradoxical property that velocity and momentum point in opposite directions.

**Problems with particles having negative energy**

Particles with negative energy, as predicted by the Dirac equation, pose two serious problems that need to be addressed:

(a) As outlined above, the velocity and the momentum vector of a particle with negative energy point in opposite directions. Therefore, if a particle with negative energy is absorbed, it transfers a momentum opposite to its own flight direction, contradicting with our everyday experience.

(b) Systems with an unlimited negative energy are thermodynamically unstable. Consider for example an atom emitting light: if the temperatures low enough, light will be emitted until all electrons are in their ground state of lowest energy. However, if we had electrons with negative energy, there would be no such ground state, meaning that the atom could emit an infinite amount of light.

**Dirac hole theory**

The possible way out of this dilemma is offered by the *Dirac hole theory*. The idea is that at zero temperature any physical system selects by itself the state of lowest energy. However, in a system with negative energy modes, the lowest energy is obtained if all modes with negative energy are occupied. This led to the concept of the so-called *Dirac sea*, in which all one-particle states with negative energy are occupied.

This concept leads to an entirely new interpretation:

- The Dirac sea characterizes the vacuum state and is physically unobservable.
- The total energy of the sea is set to zero.
- The absence of a particle with negative energy and negative momentum is inter-
interpreted as the presence of an anti-particle with positive energy and positive momentum. Note that now the velocity and the momentum point in the same direction, resolving the inconsistencies mentioned above.

- If the particles carry an additional electrical charge, the corresponding anti-particles carry the opposite charge. For example, the antiparticles of electrons ($e^-$) are called positrons ($e^+$) since they carry a positive charge.

- The bare Dirac equation describes a two-particle theory (particle and antiparticle). However, Dirac’s hole theory implicitly uses the scenario of a multi-particle theory obeying the Pauli principle. This means that this theory can only be formulated consistently in the framework of a quantum field theory with many particles.

- Particles and antiparticles recombine to radiation, for example, $e^+ e^- \rightarrow 2\gamma$.

- The opposite reaction $\gamma \rightarrow e^+ e^-$ is only possible in the presence of an atom, for example a crystal lattice, provided that the energy of the photon is high enough.

**Feynman-Stückelberg interpretation**

In the *Feynman-Stückelberg interpretation* a state of negative energy is interpreted as a state of positive energy, an inverse charge, a reflected space, and a reverse time direction, meaning that it is moving *backward in time*. The wave function of an electron with negative energy therefore corresponds to the wave function of a positron, which moves backwards in time in a reflected space.

Between electron and positron there is also the so-called *CPT symmetry*. This transformation consists of the three individual transformations in which the charge, parity and time direction are reversed. In contrast to other fundamental equations, the Dirac equation has the individual symmetries, as well as their combinations, such as CP.

### 6.2.6. Probability conservation

Now we would like to define a probability current of the Dirac wave function in analogy to the Schrödinger equation. To this end we first write down the Dirac equation

$$\left( i\gamma^\mu \tilde{\partial}_\mu - M \right) \psi = 0. \quad (6.101)$$

where we put an arrow above the $\partial$ in order to indicate that it is acting to the right on the function $\psi$. Next we write down the *adjoint Dirac equation*:

$$\psi^\dagger \left( -i(\gamma^\mu)^\dagger \tilde{\partial}_\mu - M \right) = 0. \quad (6.102)$$
Here \( \psi^\dagger \) is the adjoint spinor that may be thought of as a 4-row-vector with complex-conjugated entries. Moreover, we put an arrow pointing to the left above the \( \partial \) to indicate that it is acting to the left on \( \psi^\dagger \).

For constructing a probability density the appearance of the adjoint \( \gamma \)-matrices is somewhat unfortunate since in the end we would like to get something like \( \psi^\dagger (\ldots) \psi = 0 \). However, checking the three representations given above (Dirac, Weyl, and Majorana) we find that:

\[
\gamma^0 = (\gamma^0)^\dagger, \quad \gamma^k = -(\gamma^k)^\dagger \quad (k = 1, 2, 3)
\]

(6.103)

At first glance it seems that this minus sign can be taken into account by using the metric tensor but for this we would have to lower the index, which does not happen in this case. Fortunately, there is another easy way to get the minus, namely, by using the matrix \( \gamma^0 \):

\[
(\gamma^\mu)^\dagger = \gamma^0 \gamma^\mu \gamma^0
\]

(6.104)

Inserting this relation we can bring the adjoint equation into the original form:

\[
\begin{align*}
\psi^\dagger (-i\gamma^0 \gamma^\mu \gamma^0 \partial_\mu - M) &= 0 \\
\Rightarrow \psi^\dagger (i\gamma^0 \gamma^\mu \gamma^0 + M) &= 0 & | \gamma^0 \gamma^0 = -\eta^{00} = 1 \\
\Rightarrow \psi^\dagger (i\gamma^0 \gamma^\mu \gamma^0 + \gamma^0 M \gamma^0) &= 0 \\
\Rightarrow \psi^\dagger \gamma^0 (i\gamma^\mu \gamma^0 + M) \gamma^0 &= 0
\end{align*}
\]

(6.105)

Now we are almost there. Defining the so-called adjoint spinor

\[
\bar{\psi} := \psi^\dagger \gamma^0
\]

(6.106)

the adjoint Dirac equation reads

\[
\bar{\psi} (i\gamma^\mu \gamma^0 + M) = 0.
\]

(6.107)

**Probability current**

Now we multiply Eq. (6.101) from the left with \( \bar{\psi} \), likewise we multiply Eq. (6.107) from the right with \( \psi \), and finally we add the two equations, giving

\[
\bar{\psi} \gamma^\mu (\gamma^0 \partial_\mu + \gamma_\mu) \psi = 0.
\]

(6.108)

Here one of the derivative operators acts to the left on \( \bar{\psi} \) while the other one acts to the right on \( \psi \). Therefore, using the product rule, we can write them as a single derivative in front:

\[
\partial_\mu (\bar{\psi} \gamma^\mu \psi) = 0.
\]

(6.109)

\footnote{Note that this relation does not follow exclusively from the anticommutation relations, it is rather imposed as an additional requirement. All representations discussed here obey this property.}
This has the structure of the continuity equation with the conserved current

\[ j^\mu := \bar{\psi} \gamma^\mu \psi = \left( \rho \right) \left( \frac{1}{c^2} \right) . \]  

(6.110)

In terms of the probability density

\[ \rho = \bar{\psi} \gamma^0 \psi = \psi^4 \]  

(6.111)

and the probability 3-current \( \vec{j} \) with

\[ (\vec{j})_k = c \bar{\psi} \gamma^k \psi = c \psi^* \gamma^0 \gamma^k \]  

(6.112)

this continuity equation can be written in the usual form as

\[ \partial_t \rho = -\vec{\nabla} \cdot \vec{j} . \]  

(6.113)

**Action for the uncharged Dirac equation**

With the adjoint spinor \( \bar{\psi} = \psi^* \gamma^0 \) the appropriate classical *action* for the Dirac field can be guessed easily:

\[ S = \int d^4 x \ \bar{\psi}(x) (i \gamma^\mu \partial_\mu - M) \psi(x) . \]  

(6.114)

Note that this action is automatically a Lorentz scalar because \( \bar{\psi} \gamma^\mu \psi \), which is a contravariant Lorentz 4-vector (see exercise) is contracted with the covariant Lorentz 4-vector \( \partial_\mu \). Hence the action is invariant under Lorentz transformations.

Opposed to the action of the scalar field, which was quadratic in \( \phi \), this action is *linear* in \( \psi \) and \( \bar{\psi} \) which are seen here as separate fields. Variation with respect to \( \bar{\psi} \) yields

\[ \delta S = \int d^4 x \ \delta \bar{\psi}(x) \left( i \gamma^\mu \partial_\mu - M \right) \psi(x) , \]  

(6.115)

giving the ordinary Dirac equation

\[ (i \gamma^\mu \partial_\mu - M) \psi(x) = 0 . \]  

(6.116)

Similarly, variation with respect to \( \psi \) gives

\[ \delta S = \int d^4 x \ \bar{\psi}(x) \left( i \gamma^\mu \partial_\mu - M \right) \delta \psi(x) , \]  

(6.117)

Here we have to take care since the derivative \( \partial_\mu \) still acts to the right. In order to convert it into a derivative acting to the left, we have to partially integrate the action, omitting the boundary terms were the variation is supposed to vanish. The result reads

\[ \delta S = \int d^4 x \ \bar{\psi}(x) \left( -i \gamma^\mu \partial_\mu - M \right) \delta \psi(x) , \]  

(6.118)
giving the adjoint equation
\[ \bar{\psi}(x) \left( i\gamma^\mu \partial_\mu + M \right) = 0. \] (6.119)

### 6.2.7. Adding electromagnetic interactions

The Dirac equation *per se*, which was invented as a theory for the electron, actually does not know anything about electrodynamics or electric charges. In fact, we can find representations of the gamma matrices were the Dirac equation is entirely real-valued. An example is the Majorana representation in Eq. (6.49) on page 124. Inserting this equation into the Dirac equation \( (i\gamma^\mu \partial_\mu - M)\psi = 0 \) and writing it out in components we get

\[
\begin{pmatrix}
-\partial_1 & \partial_3 & 0 & \partial_0 - \partial_2 \\
\partial_3 & \partial_1 & \partial_0 + \partial_2 & 0 \\
0 & \partial_0 + \partial_2 & -\partial_1 & \partial_3 \\
-\partial_0 - \partial_2 & 0 & \partial_3 & \partial_1
\end{pmatrix} \psi = M\psi. \] (6.120)

This is only one example of a real-valued representation. There are actually infinitely many of them, which differ by orthogonal transformations. The fact that the Dirac equation can be made an entirely real tells us that the particle that it describes is electrically neutral.

**Remember:** The Dirac equation as it is describes electrically neutral particles with spin.

### Complexification

Of course, we would like to give the electron a charge and let it interact with the surrounding electromagnetic field. In the previous chapter we learned how to do this in a consistent way without violating gauge and variance. We showed that it boils down to

- complexifying the wave function \( \psi(x) \) in order to implement a \( U(1) \)-circle in each point of space-time, and
- replacing the ordinary derivative by the so-called covariant derivative (see Eq. (5.105) on page 110)

\[ \partial_\mu \rightarrow D_\mu = \partial_\mu - \frac{ie}{\hbar}A_\mu, \] (6.121)

where \( e \) is the elementary charge of the electron, so that the resulting Dirac equation reads:

\[ \left( i\gamma^\mu \left( \partial_\mu - \frac{ie}{\hbar}A_\mu \right) - M \right) \psi = 0. \] (6.122)

As discussed in the previous chapter, the covariant derivative is the “correct” derivative which generates not only translations in space but also the correct translations within the \( U(1) \)-degrees of freedom. In the literature the replacement rule in (6.121) is known as minimal coupling. In relativistic quantum mechanics, were \( p_\mu = -i\hbar \partial_\mu \), minimal coupling can be expressed equivalently as replacing \( p_\mu \rightarrow p_\mu - eA_\mu \). Thus, in terms of

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5This is the reason why you find various different definitions of the Majorana matrices in the literature.
the momentum the Dirac equation in a magnetic field can be written as

$$\left( \gamma^\mu (p_\mu - eA_\mu) + mc \right) \psi = 0, \quad (6.123)$$

where $mc = \hbar M$.

**Charged Dirac equation: Probability current**

The probability current for the Dirac equation in the electromagnetic field can be calculated in exactly the same way as before. We first determine the adjoint equation, getting

$$\bar{\psi} \left( i \gamma^\mu \left( \partial_\mu + \frac{ie}{\hbar} A_\mu \right) + M \right) = 0. \quad (6.124)$$

Notice that the electromagnetic term in the brackets has changed sign because of the imaginary unit. Then we multiply the original Dirac equation from the left with $\bar{\psi}$ and likewise the adjoint equation from the right with $\psi$, adding the two resulting equations. When adding the two equations, not only the mass term but also the electromagnetic term drops out, hence we stay with the previous definition of the conserved probability current, namely:

$$\partial_\mu j^\mu = 0, \quad j^\mu = \bar{\psi} \gamma^\mu \psi. \quad (6.125)$$

**Action for the charged Dirac equation**

In order to account for electromagnetism we simply have to replace the derivative in equation (6.114) by the covariant derivative. In addition we have to include the contribution of the electromagnetic field itself:

$$S = \int d^4x \left[ \bar{\psi}(x) \left( i \gamma^\mu \left( \partial_\mu - \frac{ie}{\hbar} A_\mu \right) - M \right) \psi(x) - \frac{1}{4\mu_0} F_{\mu\nu} F^{\mu\nu} \right]. \quad (6.126)$$

This action describes the full interplay between electrons and photons. Varying with respect to $\bar{\psi}$ gives the Dirac equation in the electromagnetic field

$$\left( i \gamma^\mu \left( \partial_\mu - \frac{ie}{\hbar} A_\mu \right) - M \right) \psi = 0. \quad (6.127)$$

On the other hand, variation with respect to the electromagnetic potential gives the Euler Lagrange equations for the electromagnetic field itself

$$\frac{\partial L}{\partial A_\mu} - \partial_\nu \frac{\partial L}{\partial (\partial_\nu A_\mu)} = 0 \quad (6.128)$$

or explicitly

$$\partial_\nu F^{\nu\mu} = -\mu_0 \frac{e}{\hbar} \bar{\psi} \gamma^\mu \psi. \quad (6.129)$$

As we have shown above, this current is conserved, reflecting charge conservation.
6.2 Dirac equation

**Charge conjugation**

Charge conjugation refers to a reflection of the U(1)-circle electrodynamics. On the level of the wave function charge conjugation is usually carried out by taking the complex conjugate (not to be confused with the hermitean conjugate):

\[ \psi(x) \rightarrow \psi^*(x). \]  \hspace{1cm} (6.130)

How does the Dirac equation behave under complex conjugation? Taking the complex conjugate of \( (i \gamma^\mu \partial_\mu - M) \psi = 0 \) we get

\[ (-i (\gamma^\mu)^* \partial_\mu - M) \psi^* = 0. \]  \hspace{1cm} (6.131)

Thus, if \( \psi(x) \) it is solution of the Dirac equation with \( \gamma^\mu \), then \( \psi^*(x) \) is a solution of the Dirac equation with complex conjugate matrices \( (-\gamma^\mu)^* \). Obviously, the complex conjugate matrices obey the same commutation relations as the original matrices. This means that we can differ only by a unitary transformation \( C \) in spinor space:

\[ (-\gamma^\mu)^* = C^\dagger \gamma^\mu C. \]  \hspace{1cm} (6.132)

Let us now define the charge conjugate of a Dirac spinor \( \psi \) as

\[ \psi^{(C)} := C \psi^*. \]  \hspace{1cm} (6.133)

It can be shown that \( \psi^{(C)} \) is again a spinor, that is, it transforms in the same way as an ordinary spinor under Lorentz transformations (see exercises):

\[ \psi \rightarrow S \psi \Rightarrow \psi^{(C)} \rightarrow S \psi^{(C)} \]  \hspace{1cm} (6.134)

**6.2.8. Other symmetries**

**Helicity**

Helicity is the projection of the spin onto the direction of momentum (see exercise). The helicity of a particle is right-handed if the direction of its spin is the same as the direction of its motion and left-handed if opposite. For spin-\( S \) particles it is quantized by \( -S, -S+1, \ldots, S-1, S \). For massless particles the helicity is conserved under frame changes.

**Chirality**

Chirality and helicity are similar. They are identical for massless particles while they differ in the massive case. Compared to helicity, chirality is a more abstract property: It detects whether a physical object transforms according to a right-handed or a left-handed representation of the Poincaré algebra.

In Sect. 6.2.2 on page 122 we already introduced the additional \( \gamma \)-matrix

\[ \gamma^5 := i \gamma^0 \gamma^1 \gamma^2 \gamma^3. \]  \hspace{1cm} (6.135)
This matrix is hermitean, its square is $\mathbb{1}$, and it anticommutes with all other $\gamma$-matrices, independent of the chosen representation. Hence it can have only two possible eigenvalues, namely, $\pm 1$, allowing us to define two orthogonal projection operators

$$P_\pm := \frac{1}{2}(1 \pm \gamma^5)$$

(6.136)

obeying

$$P_\pm^2 = P_\pm, \quad P_+ P_- = 0, \quad \text{and} \quad P_+ + P_- = 1.$$

In the special case of massless fermions one can easily show that if the spinor $\psi$ is a solution of the massless Dirac equation

$$\gamma^\mu \partial_\mu \psi = 0,$$  

(6.137)

then the so-called chiral spinors

$$\psi_R = \psi_+ := P_+ \psi, \quad \psi_L = \psi_- := P_- \psi$$

(6.138)

are also solutions of the Dirac equation.

**Proof:**

$$\gamma^\mu \partial_\mu \psi_\pm = \frac{1}{2} \sum_{\alpha=0}^3 \gamma^\mu \partial_\mu \psi \pm \frac{1}{2} \sum_{\alpha=0}^3 \gamma^5 \partial_\mu \psi = \pm \frac{1}{2} \sum_{\alpha=0}^3 \gamma^\mu \partial_\mu \psi = 0.$$

The solution $\psi_+$ is called a right-handed spinor, the solution $\psi_-$ correspondingly a left-handed spinor. Both solutions form two irreducible representations of the Lorentz group.

The most suitable representation for chiral fermions is the Weyl representation (which is also called the chiral representation, in which the $\gamma^5$-matrix is already diagonal:

$$\gamma^5_{\text{Weyl}} = \begin{pmatrix} -\mathbb{1} & \mathbb{1} \\ \mathbb{1} & -\mathbb{1} \end{pmatrix}.$$  

(6.139)

The upper two components represent the left-handed fermion while the lower two components represent the right-handed fermion. More specifically, the Dirac equation in the chiral representation reads:

$$(i\gamma^\mu \partial_\mu - M)\psi = \begin{pmatrix} \begin{pmatrix} -M \\ i(\partial_0 - \bar{\sigma} \cdot \nabla) \end{pmatrix} \\ \begin{pmatrix} i(\partial_0 + \bar{\sigma} \cdot \nabla) \\ -M \end{pmatrix} \end{pmatrix} \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = 0,$$

(6.140)

where $\bar{\sigma}$ is the 3-vector consisting of the Pauli matrices $\sigma^x, \sigma^y, \sigma^z$. As can be seen, the two Lorentz group representations $\psi_L$ and $\psi_R$ are mixed by the mass term in the Dirac equation. Only in the massless case $M = 0$, the equations for $\psi_L$ and $\psi_R$ decouple:

$$i(\partial_0 - \bar{\sigma} \cdot \nabla)\psi_L = 0,$$

$$i(\partial_0 + \bar{\sigma} \cdot \nabla)\psi_R = 0$$

(6.141)

These equations are called the Weyl equations. They are particularly useful when treating neutrinos and weak interactions.

---

6 An irreducible representation cannot be decomposed into a direct sum of smaller representations, meaning that it cannot written in a block-diagonal form.
Remember: The Weyl representation of the Dirac matrices is particularly suitable in the case of massless fermions.
7. Hamilton formalism

Lagrange is fine. Why do we need another formalism?

There are several reasons why the Hamilton formalism is exciting. On the one hand, it provides a different viewpoint since the equations of motion in the Hamilton formalism are first-order differential equations. On the other hand, the Hamilton formalism is much closer to quantum mechanics. In fact, it can be seen as the lowest-order (namely, classical) approximation of quantum mechanics. Having formulated a problem in the Hamilton formalism, it is quite straightforward to quantize it, e.g. when we replace Poisson brackets by commutators. One of the most important open problems in Physics is the quantization of gravity (general relativity). Therefore, from this broader perspective, is a rewarding intellectual investment to learn more about the Hamilton formalism and in particular about the role of constraints.

7.1. Non-relativistic Hamilton formalism

7.1.1. Hamilton formalism in Newtonian mechanics revisited

Hamilton function

Before we turn to the relativistic point particle, let us briefly recall the Hamilton formalism in the good old non-relativistic Newtonian mechanics. Starting point is the action

\[ S = \int L(q, \dot{q}) \, dt, \]

where the stationarity of the action under variation leads us to the usual Lagrange equation of motion

\[ \dot{p} = \frac{\partial L}{\partial \dot{q}}, \]

where

\[ p = \frac{\partial L(q, \dot{q})}{\partial \dot{q}} \]

(7.1)

is the conjugate momentum. The idea of the Hamilton formalism is to switch from the variables \((q, \dot{q})\), which depend on each other by taking the derivative, to the variables \((q, p)\). Initially both variables are considered as being independent so that finally their mutual dependence is encoded in another equation of motion. The change of variables \(\dot{q} \to p\) is carried out via a so-called Legendre transformation (see Appendix C on page 197) by defining the Hamilton function, called Hamiltonian

\[ H(q, p) = p\dot{q} - L(q, \dot{q}). \]

(7.2)
Technically the computation of the Hamiltonian requires to eliminate $\dot{q}$ as follows:

- Eq. (7.1) gives us the conjugate momentum $p$ as a function of $q$ and $\dot{q}$, for example, $p = m\dot{q}$ for a free particle.

- The first thing to do is to invert this relation, expressing $\dot{q}$ as a function of $q$ and $p$, for example, $\dot{q} = p/m$ for a free particle.

- This inverted relation $\dot{q}(q, p)$ is used to define the Hamiltonian by

$$H(p, q) = p\dot{q}(p, q) - L(q, \dot{q}(p, q)).$$  \[7.3\]

In general, if $L = T - V$, the Hamiltonian turns out to be the energy $H = T + V$.

**Example:** The second-simplest example is a non-relativistic particle in a potential with the Lagrange function $L(q, \dot{q}) = \frac{1}{2}m\dot{q}^2 - V(q)$. Here the conjugate momentum is $p = \frac{\partial L}{\partial \dot{q}} = m\dot{q}$.

Inverting this relation we get $\dot{q} = p/m$. Hence the Hamiltonian reads

$$H(q, p) = p\dot{q} - L(q, \dot{q}) = \frac{p^2}{2m} - L(q, p/m) = \frac{p^2}{2m} - \frac{p^2}{2m} + V(q) = \frac{p^2}{2m} + V(q).$$

### Equations of motion

To obtain the equations of motion in the Hamilton formalism, let us again consider the variation of the action $\delta S = \int \delta L \, dt$ with

$$\delta L(q, \dot{q}) = \frac{\partial L}{\partial q} \delta q + \frac{\partial L}{\partial \dot{q}} \delta \dot{q} = \frac{\partial L}{\partial q} \delta q + p \delta \dot{q}. \tag{7.4}$$

Because of the product rule $\delta(p\dot{q}) = \dot{q} \delta p + p \delta \dot{q}$ we can rewrite this as

$$\delta L = \frac{\partial L}{\partial q} \delta q + \delta(p\dot{q}) - \dot{q} \delta p \tag{7.5}$$

and, bringing $\delta(p\dot{q})$ to the left side, we get

$$\delta L - \delta(p\dot{q}) = \frac{\partial L}{\partial \dot{q}} \delta q - \dot{q} \delta p \quad \Rightarrow \quad \delta H = \delta(p\dot{q} - L) = \dot{q} \delta p - \frac{\partial L}{\partial q} \delta q. \tag{7.6}$$

This has to be compared with

$$\delta H = \frac{\partial H}{\partial q} \delta q + \frac{\partial H}{\partial p} \delta p, \tag{7.7}$$

giving

$$\frac{\partial H}{\partial p} = \dot{q}, \quad \frac{\partial H}{\partial q} = -\frac{\partial L}{\partial q}. \tag{7.8}$$

Plugging in the Lagrange equations of motion $\frac{\partial L}{\partial \dot{q}} = \frac{d}{dt} \frac{\partial L}{\partial q} = \dot{p}$ we arrive at the Hamilton equations of motion

$$\dot{q} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial q}. \tag{7.9}$$
Thus, instead of a single second-order differential equation, we get here two first-order differential equations. The first one restores the dependency between velocity and momentum while the second one resembles the original Lagrange equation of motions.

**Example:** In the example of the free particle given above, the equations of motions simply read \( \ddot{q} = \frac{p}{m} \) (restoring the dependency \( p = mv \) between momentum and velocity) and \( \dot{p} = -\frac{dV}{dq} \), which is equivalent to the Lagrange equation of motion \( m\ddot{q} = -\frac{dV}{dq} \).

**Phase space interpretation**

In the Hamilton formalism for the non-relativistic point particle, the solutions are parameterized as a function of time. Note that the Hamilton function \( H = T + V \) is just the energy of the system, hence it can be interpreted as the generator of translations in time.

In classical mechanics, the variables \( q \) and \( p \) are usually viewed as the coordinates of the phase space of the system. In this space, the initial condition is just a single point. As time proceeds, this point moves in phase space according to the equations of motion, producing a certain trajectory (also called orbit) in phase space. Note that the equations of motion can be written in a vectorial form as

\[
\frac{d}{dt} \begin{pmatrix} q(t) \\ p(t) \end{pmatrix} = \begin{pmatrix} \partial_p H \\ -\partial_q H \end{pmatrix}. \tag{7.10}
\]

Notice that on the right we do not have the usual gradient

\[
\nabla = \begin{pmatrix} \partial_q \\ \partial_p \end{pmatrix},
\]

instead the two components have been exchanged and the second one comes with a minus sign. In fact, we can fix this by writing

\[
\frac{d}{dt} \begin{pmatrix} q(t) \\ p(t) \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \begin{pmatrix} \partial_q \\ \partial_p \end{pmatrix} H = S \nabla H, \tag{7.11}
\]

where \( S \) is a matrix reflecting the so-called symplectic structure of classical mechanics.

This allows us to understand the Hamilton equations of motion as follows. The Hamilton function associates with each point in phase space a certain energy, and of course this energy is expected to be conserved along the actual trajectory. In order to identify the direction (the tangent vector) of the trajectory in a given point, we first compute the gradient \( \nabla H \) of the energy, as shown in the figure. This gradient vector is expected to be locally orthogonal on the manifold of constant energy. Hence, in order to follow this orbit, we need a tangent vector that is orthogonal on the
gradient. In two dimensions this vector can be constructed by changing the components and adding the minus sign in one of the components. In other words, the matrix $S$ converts the vector $\nabla H$, which is orthogonal on the equal-energy-plane, to a perpendicular vector which is tangent to this plane. This provides an easy explanation of the equations of motion in the Hamilton formalism.

**Time evolution of functions and conserved quantities**

Let $f(q,p,t)$ be a scalar function defined on phase space. What is the rate of change of this function experienced by a particle? Obviously, this change is given by the total derivative that can be calculated by the chain rule

$$\frac{d}{dt} f(q(t), p(t), t) = \frac{\partial f}{\partial q} \dot{q} + \frac{\partial f}{\partial p} \dot{p} + \frac{\partial f}{\partial t}. \quad (7.12)$$

Inserting the Hamilton equations of motion yields

$$\frac{d}{dt} f(q(t), p(t), t) = \frac{\partial f}{\partial q} \frac{\partial H}{\partial p} - \frac{\partial f}{\partial p} \frac{\partial H}{\partial q} + \frac{\partial f}{\partial t}. \quad (7.13)$$

In classical mechanics, this particular combination of derivatives on the right side is so important that one introduces a special notation, namely, the so-called poisson brackets:

$$\{f, g\} := \frac{\partial f}{\partial q} \frac{\partial g}{\partial p} - \frac{\partial f}{\partial p} \frac{\partial g}{\partial q}. \quad (7.14)$$

With this notation, the change of the function can be expressed as

$$\frac{d}{dt} f = \{f, H\} + \frac{\partial f}{\partial t}. \quad (7.15)$$

In particular, a quantity that is conserved along the trajectory of the particle and does not explicitly depend on time is characterized by a vanishing Poisson bracket.

Note that the Poisson bracket formula also captures the correct time evolution of the variables $q$ and $p$: they just reproduce the Hamilton equations of motion:

$$\dot{q} = \{q, H\} = \frac{\partial H}{\partial p}, \quad \dot{p} = \{p, H\} = -\frac{\partial H}{\partial q}. \quad (7.16)$$

**To calculate the time derivative of an object we simply have to put the object in a Poisson bracket with the Hamiltonian.**

More generally, in a system with several positions and momenta, the Poisson bracket is defined as

$$\{f, g\} := \sum_k \left( \frac{\partial f}{\partial q_k} \frac{\partial g}{\partial p_k} - \frac{\partial f}{\partial p_k} \frac{\partial g}{\partial q_k} \right). \quad (7.17)$$

It is worth being noted that the so-called fundamental Poisson brackets of the coordinates and momenta themselves are given by

$$\{q_j, q_k\} = \{p_j, p_k\} = 0, \quad \{q_j, p_k\} = \delta_{jk}. \quad (7.18)$$
Remark: The antisymmetric Poisson bracket is the counterpart of the commutator in quantum mechanics. The notation is a bit unfortunate insofar as the Poisson bracket uses curly brackets, just the same type of brackets used in quantum mechanics for the symmetric anticommutator. Both types of brackets must not be confused.

7.1.2. Constraints in the non-relativistic Lagrange formalism

A dynamical constraint is a condition that restricts the degrees of freedom, changing the solutions of the equations of motion. One distinguishes between constraints and constraining forces. Constraining forces arise through physical interactions that change or restrict the possible trajectories. For example, a ship floats on the water surface because it experiences buoyancy under water, but not above the water. Constraints, on the other hand, are geometric in nature. Here, the freedom of movement is restricted by imposing an additional equation. A well-known example is a bead that glides on a wire without friction. The bead can move in the direction of the wire, but not perpendicular to it. In this respect, a constraint is different from a constraining force, because the ship of which we have spoken is in principle free to be pushed below or above the surface of the water.

Constraints are usually given in the form of equations. If this equation depends only on the coordinates and time (and not on the momenta), and if it can be written in the functional form

\[ \chi(q_1, q_2, \ldots, t) = 0, \]  

then the constraint is called holonomic. In the Lagrange formalism such holonomic constraints can be taken into account by choosing appropriate coordinates on the constrained manifold. This is usually the most efficient approach. Here, however, we want to see how a constraint can be taken into account by introducing a Lagrange multiplier. To this end we first revisit the method of Lagrange multipliers in the Lagrange formalism.

Method of Lagrange multipliers

Consider a mechanical system with \( n \) degrees of freedom \( q_1, \ldots, q_n \), described by a Lagrange function \( L(q_1, \ldots, q_n; \dot{q}_1, \ldots, \dot{q}_n) \). Suppose that these degrees of freedom are restricted by a holonomic constraint

\[ \chi(q_1, \ldots, q_n) = 0, \]  

which, for the sake of simplicity, does not depend on time. We may think of this constraint as manifold on which the particle is bounded to glide in a frictionless manner. As such, the constraint exerts a constraining force on the particle which keeps it on the

\[ ^1 \text{Non-holonomic constraints include all those that can not be formulated as holonomic constraints. These}
\] include, for example, constraints expressed by inequalities. For example, a chair cannot follow the gravitational field and fall all the way into the center of the our planet because it is firmly standing on the surface of the earth, but it can be lifted, so its \( z \) coordinate is constrained by the inequality \( z \geq 0 \). Furthermore, the non-holonomic constraints include differential phenomena depending on the velocity such as friction.
manifold. As the particle glides frictionless, this constraining force is expected to be perpendicular on the constraining manifold.

What can we say about the constraining force? We do not yet know its magnitude, but at least we know its direction, namely, it is perpendicular on the manifold to which the motion is constrained. This direction is given by the gradient \( \nabla \chi \) of the function \( \chi \). The constraining force is oriented in this or the opposite direction with a yet unknown proportionality factor:

\[
\vec{F}_{\text{constraint}} = \lambda \nabla \chi(q_1, \ldots, q_n).
\] (7.21)

This proportionality factor \( \lambda \) is the so-called Lagrange multiplier associated with the constraint. Note that there is some kind of gauge freedom here: if we replace the function \( \chi \) by \( 2\chi \), the manifold to which the motion of the particle constrained remains of course exactly the same. However, the gradient is twice as large, implying that the Lagrange multipliers is smaller by a factor of \( \frac{1}{2} \).

**Remember:** The Lagrange multiplier associated with a constraint can be thought of as a proportionality factor relating the gradient of the constraining function with the actual constraining force. It compensates the "gauge freedom" in choosing the length of the gradient.

How can we implement the concept of the Lagrange multiplier in the Lagrange formalism? This is actually very simple: we just add the constraint times \( \lambda \) by introducing a modified Lagrange function

\[
\tilde{L}(q_1, \ldots, q_n; \dot{q}_1, \ldots, \dot{q}_n; \lambda) := L(q_1, \ldots, q_n; \dot{q}_1, \ldots, \dot{q}_n) + \lambda \chi(q_1, \ldots, q_n). \] (7.22)

In this modified Lagrangian, the Lagrange multiplier \( \lambda \) enters as an additional degree of freedom that has to be determined. Note that its temporal derivative \( \dot{\lambda} \) does not occur, hence the corresponding equation of motion

\[
\frac{\partial \tilde{L}}{\partial \lambda} = \chi(q_1, \ldots, q_n) = 0
\] (7.23)

just reproduces the constraint. The other equations of motion get an additional term

\[
\frac{d}{dt} \frac{\partial \tilde{L}}{\partial \dot{q}_k} = \frac{\partial L}{\partial q_k} + \lambda \frac{\partial \chi}{\partial q_k}
\] (7.24)

which can be interpreted as the constraining force.

**A simple example: Motion on a circle**

To start with, let us consider a free non-relativistic particle in two space dimensions, characterized by the coordinates \( x, y \) and velocities \( \dot{x}, \dot{y} \). The Lagrangian of the system is \( L = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) \) and the conjugated momenta \( p_x = m\dot{x} \) and \( p_y = m\dot{y} \) obey the equations of motion \( p_x = \dot{p}_x = 0 \), i.e., the both momenta are conserved individually.

Let us now constrain the motion of the particle onto a circle, as expressed by the holonomic constraint

\[
\chi(x, y) = x^2 + y^2 - 1 = 0.
\] (7.25)
Then the modified Lagrangian would read
\[ L \rightarrow \tilde{L} = \frac{m}{2} (x^2 + y^2) + \lambda (x^2 + y^2 - 1) \] (7.26)

for which we get three equations of motion:
\[
\begin{align*}
\frac{d}{dt} \frac{\partial \tilde{L}}{\partial \dot{x}} &= \frac{\partial L}{\partial x} \Rightarrow m\ddot{x} = 2\lambda x \\
\frac{d}{dt} \frac{\partial \tilde{L}}{\partial \dot{y}} &= \frac{\partial L}{\partial y} \Rightarrow m\ddot{y} = 2\lambda y \\
\frac{\partial L}{\partial \lambda} &= 0 \Rightarrow x^2 + y^2 = 1
\end{align*}
\] (7.27)

**Solution:**
The first two differential equations give the real solutions
\[ x(t) = A \cos(\omega t) + B \sin(\omega t), \quad y(t) = C \cos(\omega t) + D \sin(\omega t) \] (7.28)

where \( A, B, C, D \in \mathbb{R} \) are yet unknown coefficients and where
\[ \omega = \sqrt{\frac{2|\lambda|}{m}}. \] (7.29)

Inserting these solutions into the constraint yields
\[ \chi(x(t), y(t)) = \frac{1}{2} \left[ A^2 + B^2 + C^2 + D^2 - 2 + (A^2 - B^2 + C^2 - D^2) \cos(\omega t) \right. \\
\left. + 2(AB + CD) \sin(\omega t) \right] = 0 \] (7.30)

Since this has to hold for all times, we are led to three independent equations, namely
\[
\begin{align*}
A^2 + B^2 + C^2 + D^2 &= 2 \\
A^2 - B^2 + C^2 - D^2 &= 0 \\
AB + CD &= 0.
\end{align*}
\] (7.31)

Obviously we have three equations for four unknowns. The remaining degree of freedom is the initial position (angle) on the circle. The meaning of the Lagrange parameter \( \lambda = -\frac{1}{2}m\omega^2 \) is also clear: it is proportional to the magnitude of the centripetal force.

### 7.1.3. Constraints in the non-relativistic Hamilton formalism

So far we have discussed holonomic constraints in the context of the Lagrange formalism. In order to handle such constraints within the Hamilton formalism, we start from the modified Lagrangian \( \tilde{L} \) in Eq. (7.22):
\[ \tilde{L}(q_1, \ldots, q_n; \dot{q}_1, \ldots, \dot{q}_n; \lambda) := L(q_1, \ldots, q_n; \dot{q}_1, \ldots, \dot{q}_n) + \lambda \chi(q_1, \ldots, q_n). \] (7.32)
The corresponding conjugated momenta are the same as before:

\[ p_k = \frac{\partial \tilde{L}}{\partial \dot{q}_k} = \frac{\partial L}{\partial \dot{q}_k}. \]  

(7.33)

Note that the Lagrange multiplier, which enters here as a new degree of freedom, enters only in the form of a ‘coordinate’ \( \lambda \), not as a ‘velocity’ \( \dot{\lambda} \). Therefore, the conjugated momentum vanishes, i.e. \( p_\lambda = 0 \). This means that \( \lambda \) is not touched by the Legendre transformation. Hence the modified Hamiltonian reads

\[ \tilde{H}(q_1, \ldots, q_n; p_1, \ldots, p_n; \lambda) := \sum_{k=1}^{n} \dot{q}_k p_k - L(q_1, \ldots, q_n; \dot{q}_1, \ldots, \dot{q}_n) - \lambda \chi(q_1, \ldots, q_n) \]

(7.34)

with the corresponding equations of motion

\[ \dot{q}_k = \frac{\partial \tilde{H}}{\partial p_k}, \quad \dot{p}_k = -\frac{\partial \tilde{H}}{\partial \dot{q}_k}, \quad 0 = -\chi = \frac{\partial \tilde{H}}{\partial \lambda}. \]  

(7.35)

Here at the last equation again reproduces the constraint.

**How to handle the constraints**

A holonomic constraint, which depends only on the coordinates but not on the momenta, effectively reduces the number of coordinates. However, it is not clear how it reduces the number of the corresponding momenta. To this end we need additional constraints that can be obtained by iterating the given constraints, as will be described below.

Why do we need additional constraints? To understand this issue, let us consider the extreme case of \( n \) constraints of the form

\[ q_1 = q_2 = \ldots q_n = 0. \]

This set of constraints completely nails the system to the origin where all coordinates are zero. Obviously, such a system cannot move so that the conjugated momenta vanish as well. However, so far there is no equation telling us that the momenta have to vanish. We could easily start with an initial condition compatible with the constraints but with nonzero momenta. In this case the system would run out of the constraint.

This trivial example demonstrates that it is not sufficient that the system obeys the constraint initially at \( t = 0 \) but it also has to do so for all times in the future:

\[ \chi(t) := \chi(q_1(t), \ldots, q_n(t)) = 0 \quad \forall t \]  

(7.36)

This can be accounted for by Taylor-expanding the constraint

\[ \chi(t) = \chi(0) + \dot{\chi}(0) t + \frac{1}{2} \ddot{\chi}(0) t^2 + \frac{1}{6} \dddot{\chi}(0) t^3 + \ldots \]  

(7.37)

As we have seen before, the total derivatives with respect to time can be expressed
conveniently in terms of Poisson brackets. This leads us to a series of new constraints at \( t = 0 \), namely

\[
\begin{align*}
\chi &= 0 \\
\{\chi, \hat{H}\} &= 0 \\
\{\{\chi, \hat{H}\}, \hat{H}\} &= 0 \\
\{\{\{\chi, \hat{H}\}, \hat{H}\}, \hat{H}\} &= 0 \\
&\ldots
\end{align*}
\]

(7.38)

Since we cannot have infinitely many new constraints, this iteration scheme has to close at some point in the sense that it produces constraints that depend on the previous ones.

**Example: Motion on a circle**

Let us now demonstrate this constraint-generating scheme in the case of the previous example, where we considered a free particle constrained on a circle. Here the Hamiltonian is given by

\[
\hat{H}(x, y, p_x, p_y, \lambda) = \frac{2}{m}(p_x^2 + p_y^2) - \lambda(x^2 + y^2 - 1).
\]

(7.39)

We can easily write down the equations of motion. As usual, we find that the first two equation give the relation between velocity and momentum:

\[
\begin{align*}
m\ddot{x} &= p_x, \\
m\ddot{y} &= p_y.
\end{align*}
\]

(7.40)

The other two equations take the form

\[
\begin{align*}
\dot{p}_x &= -2\lambda x, \\
\dot{p}_y &= -2\lambda y,
\end{align*}
\]

(7.41)

where the yet undetermined Lagrange parameter \( \lambda \) controls the oscillation frequency. Finally, the derivative with respect to \( \lambda \) reproduces the circle constraint:

\[
\frac{\partial \hat{H}}{\partial \lambda} = \chi(x, y) = x^2 + y^2 - 1 = 0.
\]

(7.42)

To solve this system of differential equations we need appropriate initial conditions. Clearly, the initial coordinates have to obey the constraint as well, i.e., \( x \) and \( y \) have to lie on the unit circle. However, we also have to specify the initial momenta. The momenta will determine the angular velocity. But, as already mentioned above, we also have to care for the right direction of the initial momentum vector. In fact, if the initial momentum vector was not tangent to the unit circle, the particle would escape from the circle, so to say run out of the constraint, and end up in an elliptic orbit.

Following the thoughts outlined above, we compute the derivatives of the constraint
in terms of Poisson brackets:
\[
\chi = x^2 + y^2 - 1 = 0
\]
\[
\{\chi, \hat{H}\} = \frac{2}{m}(xp_x + yp_y) = 0
\]
\[
\{\{\chi, \hat{H}\}, \hat{H}\} = \frac{4}{m}(\frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \lambda(x^2 + y^2)) = 0
\]
\[
\{\{\chi, \hat{H}\}, \hat{H}\} = \frac{16}{m^2}(xp_x + yp_y) = 0
\]

Now we can easily read off the physical meaning of these constraints: The first equation \(\chi = 0\) just means that we have to move on a circle with radius 1. The second condition \(\{\chi, \hat{H}\} = 0\) implies that the momentum vector and position vector are orthogonal. Finally, the third condition \(\{\{\chi, \hat{H}\}, \hat{H}\} = 0\) tells us that \(\lambda\) equals the (negative) kinetic energy \(\lambda = -\frac{1}{2m}(p_x^2 + p_y^2)\) which is the same as \(-\frac{1}{2}m\omega^2\) on the unit circle. The last equation turns out to be proportional to the second one does not give any new information. This means that the sequence of nested Poisson brackets closes.

**Solution:** Suppose that we start with initial condition \(x(0) = 1, y(0) = 0\), and \(\dot{y}(0) = \omega > 0\). This initial state is compatible with the constraint \(\chi = 0\). The second equation in (7.43) tells us that \(p_x(0) = 0\). The third equation allows us to determine the Lagrange parameter \(\lambda = \frac{1}{2m}(p_x^2 + p_y^2) = -\frac{1}{2}m\omega^2\). With this information the equations of motion (7.41) turn into \(m\ddot{x} = -\omega^2 x\), \(m\ddot{y} = -\omega^2 y\) with the solution
\[
x(t) = \cos(\omega t), \quad y(t) = \sin(\omega t).
\]

### 7.1.4. General parameterization

Both the Lagrange formalism and the Hamilton formalism of a nonrelativistic particle use a particular parameterization, namely, time. In the nonrelativistic case, this parameterization is particularly natural. However, in a relativistic setting, where time becomes a coordinate rather than a parameter, the trajectory of a point particle has to be parameterized differently. In principle, we could use *any* parameterization (expecting the physical result to be invariant under re-parameterization), although in practice one often uses the proper time (Eigenzeit) of the particle. In the following we therefore introduce a general parameterization in the non-relativistic case in both the Lagrange and the Hamilton formalism.

**General parameterization in the Lagrange formalism**

Let us start with the simple example of a harmonic oscillator with the Lagrange function
\[
L(q, \dot{q}) = T - V = \frac{m}{2}q^2 - \frac{m}{2}\omega^2 q^2
\]
so that the action is given by
\[
S[q] = \int \left(\frac{m}{2}q^2 - \frac{m}{2}\omega^2 q^2\right) dt.
\]
Here the integration is carried out over the time \( t \), i.e., the trajectory of the particle is parameterized as a function of time. We now re-parameterize the integral by introducing a new parameter \( \tau \) in some range, where \( t(\tau) \) is some (potentially nonlinear) function. If we denote by the dot above a symbol the total derivative with respect to the new parameter \( \tau \), the re-parameterized action reads

\[
S[q] = \int \left( \frac{m q^2}{2} - \frac{m}{2} \omega^2 q^2 \right) t \, d\tau = \int \left( \frac{m q^2}{2} - \frac{m}{2} \omega^2 q^2 \right) d\tau. \tag{7.46}
\]

Now \( t \) and \( \dot{t} \) can be seen as a coordinate and its velocity on equal footing with \( q \) and \( \dot{q} \) (where the velocity is now the derivative with respect to the new parameter \( \tau \)). The new corresponding Lagrange function reads

\[
L(q, \dot{q}, t, \dot{t}) = \frac{m}{2} \dot{q}^2 \dot{t} - \frac{m}{2} \omega^2 q^2 \dot{t}. \tag{7.47}
\]

This Lagrange function allows us to define two conjugate momenta:

\[
p_q = \frac{\partial L}{\partial \dot{q}} = \frac{m \dot{q}}{\dot{t}}, \quad \quad p_t = \frac{\partial L}{\partial \dot{t}} = -\frac{m}{2} \omega^2 q^2. \tag{7.48}
\]

The new thing here is the kind of temporal momentum which, as we shall see below, can be interpreted the energy. Moreover, having two independent degrees of freedom, we get two equations of motion

\[
\frac{d}{d\tau} \frac{\partial L}{\partial \dot{q}} = \frac{\partial L}{\partial q} \quad \Rightarrow \quad \dot{p}_q = -m \omega^2 q \dot{t}
\]

\[
\frac{d}{d\tau} \frac{\partial L}{\partial \dot{t}} = \frac{\partial L}{\partial t} \quad \Rightarrow \quad \dot{p}_t = 0. \tag{7.49}
\]

**Example:** Calculating the derivatives of the momenta, the equations of motion read

\[
p_q = \frac{m \ddot{q}}{\dot{t}} - \frac{m \dot{q}^2}{\dot{t}^2} = -m \omega^2 q \dot{t}
\]

\[
p_t = -m \frac{\ddot{q} \dot{t}^2}{\dot{t}^3} + m \frac{\dot{q}^2 \dot{t}}{\dot{t}^3} - m \omega^2 q \dot{q} = 0.
\]

As a simple example let us consider the parameterization \( t = \frac{1}{2} \tau^2 \) (which is probably one of the worst choices for this type of problem). Then \( \dot{t} = \tau \) and \( \ddot{t} = 1 \). Inserting this parameterization into the first equation given above and solving it via Mathematica yields the solution

\[
q(\tau) = A \cos\left( \frac{\omega}{2} \tau^2 \right) + B \sin\left( \frac{\omega}{2} \tau^2 \right),
\]

where \( A \) and \( B \) are integration constant. This is of course what we have expected. Interestingly, inserting this solution into the second equation we find that

\[
\dot{p}_t = 0 \quad \Rightarrow \quad p_t = -\frac{1}{2} m \omega^2 (A^2 + B^2) = \text{const}.
\]

This confirms that the temporal momentum is conserved.
General parameterization in the Hamilton formalism – Lapse function

If we compute the Hamiltonian by a Legendre transformation in both variables $\dot{q}$ and $\dot{t}$, it is surprising to find that

$$H(q, p_q, t, p_t) = p_q \dot{q} + p_t \dot{p} - L = 0.$$  \hfill (7.50)

**Proof:** To prove this relation we first insert the definition of the Lagrange function

$$H = p_q \dot{q} + p_t \dot{t} - \frac{m}{2} \dot{q}^2 t - \frac{m}{2} \omega^2 \dot{q}^2.$$  \hfill (7.48)

Substituting $\frac{m}{2} \omega^2 \dot{q}^2 = -p_t - \frac{m}{2} \dot{q}^2$ from Eq. (7.48) this turns into

$$H = p_q \dot{q} + p_t \dot{t} - \frac{m}{2} \dot{q}^2 t - p_t - \frac{m}{2} \dot{q}^2 t = p_q \dot{q} - \frac{m}{2} \dot{q}^2.$$  \hfill (7.51)

With $p_q = \frac{m \dot{q}}{\dot{t}}$ from Eq. (7.48) we finally get $H = 0$. \hfill \Box

$H = 0$? The Hamiltonian vanishes? What is this? This would imply that all equations of motion are trivial and that all momenta are constant. There must be something wrong here.

And indeed, there is something wrong. The point we are missing here is that the four variables, $q, p_q, t,$ and $p_t$, are treated as independent, but in reality they are not independent. In fact, it is easy to see that the momenta (7.48)

$$p_q = \frac{m \dot{q}}{\dot{t}}$$  \hfill (7.52)

are not independent but related by

$$p_t = -\frac{m \dot{q}^2}{2 \dot{t}^2} - \frac{m}{2} \omega^2 \dot{q}^2$$

This relation is easy to understand: $p_t$ is the momentum in the direction of time, i.e. something like the energy, and on the right side we have indeed the energy of the particle up to a minus sign. The relation tells us that the energy has the correct value and that it is conserved in time. Therefore, the relation acts as a constraint, a so-called Hamiltonian constraint.

$$\chi = \left( p_t + \frac{p_q^2}{2m} + \frac{m}{2} \omega^2 \dot{q}^2 \right)$$  \hfill (7.53)

To implement the Hamiltonian constraint, we use again a Lagrange multiplier and subtract it from the Hamiltonian (which is zero), resulting in

$$H(q, p_q, t, \dot{p}, \lambda) = \lambda \chi = \lambda \left( p_t + \frac{p_q^2}{2m} + \frac{m}{2} \omega^2 \dot{q}^2 \right).$$  \hfill (7.54)
The equations of motion then read

\[ \dot{q} = \{q, H\} = \frac{\partial H}{\partial p_q}, \quad \dot{p}_q = \{p_q, H\} = -\frac{\partial H}{\partial q}, \quad (7.55) \]

\[ \dot{t} = \{t, H\} = \frac{\partial H}{\partial p_t}, \quad \dot{p}_t = \{p_t, H\} = -\frac{\partial H}{\partial t}. \]

In particular, the last two equations tell us how quickly the time elapses in the given parameterization:

\[ \dot{t} = \lambda, \quad p_t = \text{const}. \quad (7.56) \]

The Lagrange multiplier \( \lambda = \frac{dt}{d\tau} \) is said to be the lapse function. Note that the lapse function itself may depend on \( \tau \). In addition, note that it does not make sense to iterate a Hamiltonian constraint in nested Poisson brackets since \( \{H, H\} = 0 \).

**Solution:** Let us write down the first two equations of motion explicitly:

\[ \frac{dq}{d\tau} = \lambda \frac{p_q}{m}, \quad \frac{dp_q}{d\tau} = -\lambda m \omega^2 q. \quad (7.57) \]

Here we can divide by \( \lambda \) and realize that

\[ \frac{1}{\lambda} \frac{d}{d\tau} = \frac{d\tau}{dt} = \frac{d}{dt} \quad (7.58) \]

turning the equations of motion into

\[ m \frac{d}{dt} q = p_q, \quad \frac{d}{dt} p_q = m \omega^2 q. \quad (7.59) \]

These are just the ordinary equations of motion of the harmonic oscillator in the usual time parameterization that we obtain by eliminating the lapse function.

### 7.2. Relativistic Hamilton formalism

#### 7.2.1. Relativistic Hamilton mechanics of a point particle

**Hamiltonian constraint**

In the relativistic setup, this works exactly in the same way. We start from a Lagrange function \( L(x, \dot{x}) \) in a chosen parameterization and define a Hamiltonian using a Legendre transformation by

\[ H(x, p) = x \cdot p - L(x, \dot{x}). \quad (7.60) \]

where the dot denotes the temporal derivative with respect to the Eigenzeit \( \tau \). As usual, this is meant in the sense that one first inverts the definition of the conjugate momentum \( p(x, \dot{x}) = \frac{\partial L(x, \dot{x})}{\partial \dot{x}} \), expressing the velocity \( \dot{x}(x, p) \) as a function of the momentum, and inserting this inverted expression in the Legendre transform given above:

\[ H(x, p) = \dot{x}(x, p) \cdot p - L(x, \dot{x}(x, p)). \quad (7.61) \]
As an example, let us consider the free point particle with the Lagrangian

\[ L(x, \dot{x}) = -mc\sqrt{-\dot{x}^\mu \dot{x}^\nu}. \]  

(7.62)

Here the conjugated momentum is

\[ p_{\mu} = \frac{\partial L(x, \dot{x})}{\partial \dot{x}^\mu} = \frac{mc \dot{x}^\mu}{\sqrt{-\dot{x}^\mu \dot{x}^\nu}} \]  

(7.63)

This allows us to compute the Hamiltonian

\[ H = p_{\mu} \dot{x}^\mu - L = \frac{mc \dot{x}^\mu \dot{x}^\mu}{\sqrt{-\dot{x}^\mu \dot{x}^\nu}} + mc\sqrt{-\dot{x}^\mu \dot{x}^\nu} = \frac{-mc(\sqrt{-\dot{x}^\mu \dot{x}^\nu})^2}{\sqrt{-\dot{x}^\mu \dot{x}^\nu}} + mc\sqrt{-\dot{x}^\mu \dot{x}^\nu} \]  

(7.64)

\[ = -mc\sqrt{-\dot{x}^\mu \dot{x}^\nu} + mc\sqrt{-\dot{x}^\mu \dot{x}^\nu} = 0 \]  

\[ \Rightarrow \ H = 0. \]

So again the Hamiltonian is zero. This is exactly the same result as in the non-relativistic case studied above when we tried to consider time as a coordinate rather than a parameter. As in the previously studied case, we made a mistake, namely, we assumed all coordinates \( x^\mu \) and momenta \( p^\mu \) to be independent. In fact, they are not: they are constrained by the energy-momentum relation \( p^\mu p_\mu = m^2c^2 \). This can be cast as a holonomic constraint, the so-called Hamiltonian constraint

\[ \chi = p^\mu p_\mu - m^2c^2 = 0 \]  

(7.65)

and we would subtract it from the Lagrange function

\[ L \mapsto \tilde{L} = L - \lambda \chi \]  

(7.66)

translating into

\[ H \mapsto \tilde{H} = \lambda \chi. \]  

(7.67)

Now the relativistic equations of motion read

\[ \dot{x}^\mu = \frac{\partial H}{\partial p_\mu} = \lambda p^\mu, \quad \dot{p}_\mu = -\frac{\partial H}{\partial x^\mu} = 0, \quad \chi = \frac{\partial H}{\partial \lambda} = 0 \]  

(7.68)

The last equation gives just the constraint

\[ p^\mu p_\mu = \frac{1}{c^2} \sqrt{\dot{x}^\mu \dot{x}_\mu} = m^2c^2 \quad \Rightarrow \quad \lambda = \frac{1}{m}. \]  

(7.69)

reproducing the expected result

\[ p^\mu = m \dot{x}^\mu, \quad \dot{p}^\mu = 0. \]
7.2 Relativistic Hamilton formalism

7.2.2. Hamilton formalism for the electromagnetic field

Equal-time slices

Let us start with the Lagrangian of the electromagnetic field without sources (see Eq. (5.44) on page 97):

\[
\mathcal{L}(A, \nabla A) = -\frac{1}{4\mu_0}F_{\mu\nu}F^{\mu\nu} = \frac{1}{2\mu_0}\left(\frac{E^2}{c^2} - \frac{B^2}{c^2}\right).
\] (7.70)

Here we do not have an Eigenzeit parameter of a particle, instead we have to choose a certain parameterization. In a given coordinate system, it is natural to use the time coordinate \(x^0 = ct\) as quantity that slices space-time into a foliation of equal-time slices. Of course, this depends on the choice of the coordinate system, and seen from a different inertial frame the slices would still be slices, but not at equal time, as sketched in the figure.

The Hamilton formulation of Electrodynamics to be introduced below is based on a particular slicing of space-time, namely, by equal-time slices. The “coordinates” \(q_k\) are now replaced by the fields \(A^\mu(x)\). That is, we have infinitely many coordinates, namely, one in each point of space-time.

Conjugated momenta and Poisson brackets

In a Hamilton setting we need conjugated momenta. This is where the choice of the foliation enters because we consider the derivatives of the field \(A_\mu(x)\) with respect to the slicing parameter (= \(ct\)), denoted here by \(\dot{A}_\mu(x) = \partial_0 A_\mu(x)\). The conjugate momenta are defined by

\[
\pi^\mu(x) := \frac{\partial \mathcal{L}}{\partial \partial_0 A_\mu}.
\] (7.71)

The “coordinates” \(A_\mu(x)\) and the conjugated “momenta” \(\pi^\mu(x)\) are now considered as independent variables in the Hamilton formalism on a given time slice of the foliation. On this time slice, we define the Poisson bracket as the 3d-integral

\[
\{f, g\} := \int d^3z \left( \frac{\partial f}{\partial A_\mu(z)} \frac{\partial g}{\partial \pi^\mu(z)} - \frac{\partial g}{\partial A_\mu(z)} \frac{\partial f}{\partial \pi^\mu(z)} \right),
\] (7.72)

where the integration is carried out over the time slice. With this definition, the electromagnetic field and its conjugate field obey the fundamental Poisson brackets

\[
\{A_\mu(x), \pi^\nu(y)\} = \delta_\mu^\nu \delta(x - y).
\] (7.73)

Zero temporal momentum constraint

In the case of an electromagnetic field without sources, where \(\mathcal{L} = -\frac{1}{4\mu_0}F_{\mu\nu}F^{\mu\nu}\), the
conjugated momentum field is given by

\[ \pi^\mu(x) = \frac{\partial L}{\partial (\partial_0 A_\mu)} = -\frac{1}{\mu_0} F^0\mu \]  

(7.74)

Because of the antisymmetry of the field strength tensor \( F^{\mu\nu} \) we immediately find that the momentum in temporal direction vanishes:

\[ \pi^0(x) = 0. \]  

(7.75)

This relation plays the role of a constraint.

**Remark:** Let us briefly think about the meaning of this constraint. Generally, a momentum tells how strongly the action responds to the velocity of the corresponding coordinate. A vanishing momentum tells us that the action does not feel the velocity of the corresponding coordinate, it would not influence the dynamics of the other degrees of freedom.

The other components can be read off from the representation of Eq. (3.59) on page 62 in the ‘mostly plus’ convention. Here we get

\[ \pi^i(x) = -\frac{1}{\mu_0 c} E^i(x), \quad (i = 1, 2, 3) \]  

(7.76)

i.e., the electric field \( \vec{E} \) plays the role of a conjugate momentum of the vector potential \( \vec{A} \).

**Setting up the Hamiltonian**

Now it is straight forward to perform the Legendre transformation and to define the Hamilton density

\[ \mathcal{H}(A(x), \pi(x)) = \dot{A}_\mu \pi^\mu - L. \]  

(7.77)

To express \( \dot{A}_\mu = \partial_0 A_\mu \) in the first term on the right side, recall that

\[ \mu_0 \pi_i = -F^0_i = F_{0i} = \partial_0 A_i - \partial_i A_0 \quad \Rightarrow \quad \dot{A}_i = \mu_0 \pi_i + \partial_i A_0, \]  

(7.78)

hence

\[ \mathcal{H}(A(x), \pi(x)) = \frac{1}{4\mu_0} F_{\mu\nu} F^{\mu\nu} + \mu_0 \pi_i \pi^i + \left( \partial_i A_0 \right) \pi^i \]  

\[ = \frac{1}{4\mu_0} F_{ij} F^{ij} - \frac{\mu_0}{2} \pi_i \pi^i + \mu_0 \pi_i \pi^i + \left( \partial_i A_0 \right) \pi^i. \]  

(7.79)

where the sum over latin indices \( i, j, \ldots \) runs only from 1 to 3. This can be rewritten as

\[ \mathcal{H}(A(x), \pi(x)) = \frac{1}{2\mu_0} \left( \vec{B}^2 + \frac{\vec{E}^2}{c^2} \right) - \frac{1}{\mu_0 c} \left( \partial_i A_0 \right) E^i \]  

(7.80)
This is the Hamiltonian density on the time slice. To get the total Hamiltonian this density has to be integrated over the equal-time slice:

\[
H = \int d^3y \mathcal{H}(A(y), \pi(y))
\]

\[
= \int d^3y \left( \frac{1}{4\mu_0} F_{ij}(y) F^{ij}(y) + \frac{\mu_0}{2} \pi_i(y) \pi^i(y) + (\partial_i A_0(y)) \pi^i(y) \right) \tag{7.81}
\]

Here we can partially integrate the last term, where we – as usual – neglect the boundary terms:

\[
H = \int d^3y \left( \frac{1}{4\mu_0} F_{ij}(y) F^{ij}(y) + \frac{\mu_0}{2} \pi_i(y) \pi^i(y) - (\partial_i \pi^i(y)) A_0(y) \right) \tag{7.82}
\]

**Handling the constraints**

The constraint \( \chi(x) = \pi^0(x) = 0 \) is taken into account by adding a term with a Lagrange multiplier \( \lambda(x) \):

\[
H \rightarrow H' = \int d^3y \left( \frac{1}{4\mu_0} F_{ij}(y) F^{ij}(y) + \frac{\mu_0}{2} \pi_i(y) \pi^i(y) - (\partial_i \pi^i(y)) A_0(y) + \lambda(y) \pi^0(y) \right) \tag{7.83}
\]

At first we check whether the constraint \( \chi = \pi^0(x) = 0 \) is conserved in time. To this end we compute the Poisson bracket

\[
\partial_0 \chi(x) = \{ \pi^0(x), H' \}. \tag{7.84}
\]

Looking at the fundamental Poisson brackets in Eq. (7.73) it is clear that only those terms in the Hamiltonian will contribute that contain \( A_0 \) or \( \partial_i A_0 \), which is conjugate to \( \pi^0 \). Thus, only the third term in the Hamiltonian will contribute, hence

\[
\{ \pi^0(x), H' \} = - \int d^3z \frac{\partial \pi^0(x)}{\partial \pi^\mu(z)} \frac{\partial \left( - \int d^3y \left( \partial_i \pi^i(y) \right) A_0(y) \right)}{\partial A_\mu(z)}
\]

\[
= \int d^3z \int d^3y \frac{\partial \pi^0(x)}{\partial \pi^\mu(z)} \frac{\partial \left( \left( \partial_i \pi^i(y) \right) A_0(y) \right)}{\partial A_\mu(z)} \tag{7.85}
\]

\[
= \int d^3y \frac{\partial \left( \left( \partial_i \pi^i(y) \right) A_0(y) \right)}{\partial A_0(x)} = \partial_i \pi^i(x)
\]

So technically this constraint gives a another constraint, namely

\[
\chi_G = \partial_i \pi^i(x) = 0. \tag{7.86}
\]

Physically the constraint \( \partial_i \pi^i = -\vec{\nabla} \cdot \vec{E} = 0 \) is easy to interpret: It tells us that there are no electric charges (this is in fact the assumption from where we started). However, let us continue to handle it as a formal constraint. First we add this constraint to the
Hamiltonian with a Lagrange multiplier \( \gamma(x) \):

\[
H'' = \int d^3y \left( \frac{1}{4H_0} F_{ij}(y)F^{ij}(y) + \frac{H_0}{2} \pi_i(y)\pi^i(y) - \left( \partial_i \pi^i(y) \right) A_0(y) + \lambda(y)\pi^0(y) + \gamma(y)\partial_i\pi^i(y) \right)
\]

(7.87)

Obviously, this Lagrange multiplier can be absorbed in \( A_0 \):

\[
H'' = \int d^3y \left( \frac{1}{4H_0} F_{ij}(y)F^{ij}(y) + \frac{H_0}{2} \pi_i(y)\pi^i(y) - \left( \partial_i \pi^i(y) \right) (A_0(y) + \gamma(y)) + \lambda(y)\pi^0(y) \right)
\]

(7.88)

Now we test again whether the new constraint is conserved in time

\[
\partial_0(\partial_i\pi^i(x)) = \{\partial_k\pi^k(x), H''\} = \partial_k\{\pi^k(x), H''\}
\]

(7.89)

and we find that indeed

\[
\partial_0(\partial_i\pi^i(x)) = 0
\]

(7.90)

Thus, the condition \( \partial_i\pi^i(x) = 0 \) (or \( \nabla \cdot \vec{E} = 0 \)) is preserved in time and thus it is a consistent constraint.

**Proof:** In this Poisson bracket, only those terms of the Hamiltonian contribute which contain the fields conjugate to \( \pi^k(x) \), that is \( A_j(x) \). Therefore, the only contributing term is the first one on the r.h.s. and thus it suffices to check the expression

\[
\partial_k\{\pi^k(x), F_{ij}(y)F^{ij}(y)\} = ?
\]

(7.91)

Here we need an fundamental property of Poisson brackets. They act as derivatives, that is, they obey the usual product rule. For example

\[
\{f, gh\} = (f, gh) + \{f, h\}g, \quad \text{or} \quad \{f, g^2\} = 2\{f, g\}g.
\]

implying that

\[
\partial_k\{\pi^k(x), F_{ij}(y)F^{ij}(y)\} = 2\partial_k\{\pi^k(x), F_{ij}(y)\}F^{ij}(y).
\]

(7.92)

Now we can compute the Poisson bracket directly as follows:

\[
\partial_k\{\pi^k(x), H''\} = \frac{1}{4H_0} \partial_k \int d^3y \{\pi^k(x), F_{ij}(y)F^{ij}(y)\}
\]

\[
= \frac{1}{2H_0} \partial_k \int d^3y \{\pi^k(x), F_{ij}(y)\}F^{ij}(y)
\]

\[
= \frac{1}{2H_0} \partial_k \int d^3y \int d^3z \left( - \frac{\partial[\pi^k(x)]}{\partial[\pi^p(z)]} \frac{\partial[A_j(y) - \partial_jA_0(y)]}{\partial[A_p(z)]} \right) F^{ij}(y)
\]

\[
= \frac{1}{2H_0} \partial_k \int d^3y \left( - \frac{\partial[A_j(y)]}{\partial[A_0(z)]} F^{ij}(y) + \frac{\partial[A_j(y)]}{\partial[A_0(z)]} F^{ij}(y) \right)
\]

(7.93)

Here the functional derivative cannot be carried out because of the derivatives, but fortunately we can integrate both terms by parts (as usual we ignore the boundary contributions):

\[
\partial_k\{\pi^k(x), H''\} = \frac{1}{2H_0} \partial_k \left( \partial_j F^{jk}(x) - \partial_j F^{ij}(x) \right) = 0
\]

(7.94)
Equations of motion

After systematically constructing all relevant constraints, we can now derive the equations of motion. The temporal components of the coordinates $A_\mu$ and momenta $\pi_\mu$ evolve as

$$\dot{A}_0(x) = \{A_0(x), H''\} = \frac{\partial H''}{\partial \pi_0(x)} = \lambda(x).$$  \hfill (7.95)

and, of course, the temporal momentum vanishes:

$$\pi^0(x) = \pi^0(x) = 0.$$  \hfill (7.96)

We could choose $\lambda(x) = 0$, implying that $A_0 = \phi/c = \text{const}$. The spatial components $A_i$ are found to evolve as

$$\dot{A}_i(x) = \{A_i(x), H''\} = \frac{\partial H''}{\partial \pi_i(x)} = \mu_0 \left( \pi_i(x) + \partial_i(A_0(x) - \gamma(x)) \right).$$  \hfill (7.97)

Here we can choose the gauge field to be given by $\gamma(x) = A_0(x)$ so that the second term vanishes. This leaves us with $\dot{A}_i = -\frac{1}{c}E_i$ or equivalently $\partial_i A = -E$. Applying the curl $\nabla \times$ to both sides yields the Maxwell equation

$$\nabla \times E = -\frac{\partial B}{\partial t}.$$  

Finally the equation of motion for the spatial momenta render the expression

$$\dot{\pi}_k(x) = \{\pi^i(x), H''\} = -\frac{1}{\mu_0} \partial_j F_{jk}(x)$$  \hfill (7.98)

giving us the Maxwell equation

$$\nabla \times B = \frac{\partial E}{\partial t}.$$  

Proof: The proof of this relation is straight-forward:

$$\dot{\pi}_k(x) = \{\pi^i(x), H''\} = \frac{1}{4\mu_0} \int d^3y \frac{\partial[F_{ji}(y)\pi_j(y)]}{\partial[A_i(x)]}$$

$$= \frac{1}{2\mu_0} \int d^3y F_{ij}(y) \frac{\partial[A_i(y)]}{\partial[A_j(x)]}$$

$$= \frac{1}{2\mu_0} \int d^3y F_{ij}(y) \frac{\partial[A_j(y)] - \partial_i[A_j(y)]}{\partial[A_j(x)]}$$

$$= \frac{1}{\mu_0} \int d^3y F_{ij}(y) \frac{\partial[A_j(y)]}{\partial[A_i(x)]}$$  \hfill (7.99)

This can be integrated by parts, as usual neglecting the boundary terms:

$$\dot{\pi}_k(x) = \{\pi^i(x), H''\} = -\frac{1}{\mu_0} \int d^3y \partial_j F_{ij}(y) \frac{\partial[A_j(y)]}{\partial[A_i(x)]} = -\frac{1}{\mu_0} \partial_i F_{jk}$$  \hfill (7.100)

Where are the other two Maxwell equations? One is encoded in the second constraint,
telling us that we have no electric sources (charges):
\[ \partial_i \pi_i(x) = 0 \Rightarrow \vec{\nabla} \cdot \vec{E} = 0 \] (7.101)

Finally the last Maxwell equation \( \vec{\nabla} \cdot \vec{B} = 0 \) (no magnetic monopoles) is already encoded in the antisymmetry of \( F^{ij} \) or, equivalently, the relation \( \vec{B} = \vec{\nabla} \times \vec{A} \).

Note that we chose a particular gauge for the Lagrange multipliers. This gauge is known as the Weyl gauge or Hamiltonian gauge. It is a very special gauge in which the zero-component field \( A_0 \), namely, the electric potential \( \phi \), is “gauged away”.

You find it strange that it is possible to gauge an electric potential away? No, it is not. In fact, you can always gauge away one of the components in the 4-potential \( A_\mu \). To this end consider a conventional 4-dimensional gauge transformation
\[ A_\mu(x) \rightarrow \tilde{A}_\mu(x) = A_\mu(x) + \partial_\mu f(x) \] (7.102)
where \( f(x) \) is some differentiable scalar function. If we consider only the temporal part \( \tilde{A}_0 = A_0 + \partial_0 f \) and we want to gauge \( A_0 \) away, we have to find an \( f(x) \) such that \( A_0(x) + \partial_0 f(x) = 0 \). The obvious solution is
\[ f(x) = c \int dt A_0((x) + C(x) \] (7.103)
where the integration constant \( C \) is a time-independent function, showing that the above condition fixes the gauge only partially. Similarly, we could gauge away one of the spatial components, although this does not make much sense.

Towards quantum electrodynamics

Why did we do all this? Because the formulation of a problem in the Hamilton formalism allows us to quantize the problem in a straightforward manner. The idea is to

- replace the “coordinates” \( A_\mu \) by “position operators” \( \hat{A}_\mu \).
- replace the “momenta” \( \pi^\nu \) by “momentum operators” \( \hat{\pi}^\nu \).
- replace Poisson brackets by commutators \( \{f, g\} \rightarrow [\hat{F}, \hat{G}] \).
- find operator representations that obey the canonical commutation relations

\[ [\hat{A}_\mu(x), \hat{\pi}^\nu(y)] = i\hbar \delta^\nu_\mu \delta(x - y) \].

The art is to handle the constraints in a way that they remain compatible with the quantum fluctuations. The remainder is to Fourier-expand the fields and to define creation and annihilation operators. However, this is far beyond the scope of the present lecture.

The main intention why this lecture covers the Hamilton approach is to prepare for a systematic quantization procedure. Probably this will be very important in the near future in the context of the yet unknown Physics on the Planck scale.
8. Towards Gravity

The purpose of this last chapter is to give a general introduction into the concepts of General Relativity (GR), which is the currently accepted classical theory of gravity. The presentation serves only as a first glance and is on purpose somewhat superficial. If you want to learn General Relativity in more detail we recommend you to join the corresponding course in our master program. Particularly interested bachelor students can already attend this course and get a voucher (Übungsschein), which can later be credited once they are enrolled in the master program.

8.1. The concept of GR

8.1.1. Gravity as curvature of space-time

After Einstein’s seminal 1905 paper on the “Electrodynamics of moving objects”, where he described the core of the theory of Special Relativity (SR), Herrmann Minkowski and others quickly clarified the mathematical structure of the theory. It became clear that SR describes the physics of particles and fields in the absence of gravity seen from inertial frames (non-accelerated observers). In each of these frames, the physical space is modeled as a vector space with a diagonal metric tensor, which, when using the ‘mostly plus’ convention, is given by

\[
\eta_{\mu \nu} = \eta^{\mu \nu} = \begin{pmatrix}
-1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.
\] (8.1)

Although this metric tensor has a non-trivial minus sign in the temporal component, we will see that the Minkowski space is in fact a flat metric without curvature. We have already seen that transformations between different inertial frames turn out to be linear $SO^+(3,1)$-transformations, including Lorentz boosts and rotations. These transformations preserve the diagonal Minkowski metric (8.1) and map straight lines to straight lines (cf Sect. 2.2.3 on page 46).

A couple of years later, when the mathematical structure of SR was understood, it became clear that the concept of relativity, in which time as a global parameter is replaced by local time coordinate $x^0 = ct$, may lead us to a radically new type of theory of gravity. The basic idea is that the 4D space-time is not intrinsically flat but curved, and that the curvature would have a similar effect on moving bodies as Newton’s gravitational forces in a flat space.
The essential conceptual idea of GR is visualized in Fig. 8.1. The left panel shows a flat 1+1-dimensional Minkowski space with an initial condition (dashed red line) where two particles are resting at certain positions. In Minkowski space such resting particles are described by parallel world lines so that their spatial distance remains constant. Obviously, there is no gravity at work in this picture.

The right panel shows the same situation on a curved manifold, namely, on a 2D sphere. Here the longitude $\phi$ serves as a spatial coordinate $x$ while the latitude $\theta$ plays the role of time $ct$. Now, if we consider an initial state (dashed red line) near the equator with two resting particles, we expect that their longitude will not change as time proceeds. Hence the world lines of both of them are heading straight north, but in contrast to the flat space scenario, they come closer and eventually collide at the North Pole. This means that the actual metric distance between the two particles decreases as time advances. This illustrates that the curvature of the underlying space leads to an apparent attraction of the two particles.

Accepting that an underlying curvature of space-time can be interpreted as an apparent gravitational force, the next step would be to understand where this curvature comes from. As in electrodynamics, where physical twists of the $U(1)$-tunnels are caused by electric charges, we expect the curvature of space-time to be generated by the sources of gravity, i.e., by the mass and the energy of the objects living in space-time.

To summarize, GR consists of two main ideas:

- The Newtonian concept of forces is abolished. There are no forces any more. Instead particles always propagate force-free on straight lines in a curved space-time.
- The curvature of space-time is caused by the mass and energy content of the objects living in space-time.

Correspondingly GR is based on two types of equations:

- The geodesic equation tells us what it means to move straight (i.e. without acceler-
8.2 Curved space-time

• The Einstein field equations of GR explain how mass and energy generate curvature.

Both sets of equations are highly intertwined because the motion of particles changes the distribution of mass and energy and therefore back-reacts on the curvature landscape.

As we will see below, GR is essentially a gauge theory like electrodynamics, the main difference being that we are no longer dealing with distortions of internal degrees of freedom like the $U(1)$-circle but with distortions of space-time itself. In fact, as we will see, GR is based on very similar mathematical concepts. Keywords like the covariant derivative, which already sound familiar to us, play a central role in GR as well.

How can we see or feel curvature? The answer is very simple: we perceive curvature as gravity. Some people go even further in saying that gravity and space-time are identical.

In some sense there is some irony in Newton’s theory. At his time Newton was unable to realize that space-time is curved. As a way out he invented the magic concept of forces at distance which effectively mimic the influence of the curvature to lowest order.

8.2. Curved space-time

8.2.1. Position-dependent metric tensor

As already mentioned, SR takes place in Minkowski space with a constant metric $g_{\mu\nu} = \eta_{\mu\nu} = \text{diag}(-1,1,1,1)$. In GR this concept of a vector space with a constant metric does no longer work. As we will see below, a curved space-time requires that the metric tensor depends on the position in space-time. But beware; conversely, a position-dependent metric does not automatically imply that spacetime is curved, as we will demonstrate in the following:

**Flat space: Euclidean vs. polar coordinates**

As a preliminary exercise, let us study the good old Euclidean $\mathbb{R}^2$ with the standard metric $g_{\mu\nu} = \delta_{\mu\nu}$. Obviously this is a flat space like a flat piece of paper. As everyone of us knows, this space can be described by a large variety of coordinate systems, the most important one being Cartesian coordinates and polar coordinates, as illustrated in Fig. 8.2.

Note that the abstract object studied here is the Euclidean $\mathbb{R}^2$ with certain points $p \in \mathbb{R}^2$. The points can be represented uniquely by coordinates, and there are of course many possible coordinate representations. From a computational perspective, each of them has certain advantages and disadvantages. For example, in Cartesian coordinates
it is very simple to compute the distance $\Delta s$ between two points $A$ and $B$ by Pythagoras formula (cf Fig. 8.2)

$$\Delta s^2 = g_{ij} \Delta x^i \Delta x^j, \quad (8.2)$$

where $\Delta x^i = x^i_B - x^i_A$ and $g_{ij} = \delta_{ij} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ is the Euclidean metric.

In polar coordinates this is not so easy. The two points are now labeled by two radii $r_A, r_B$ and two angles $\phi_A, \phi_B$. These coordinates can be mapped back to Cartesian coordinates by

$$x^1 = r \cos \phi, \quad x^2 = r \sin \phi. \quad (8.3)$$

Using this map it is straightforward to show that the distance between the two points in polar coordinates is given by

$$\Delta s^2 = r_A^2 + r_B^2 - 2r_A r_B \cos(\phi_B - \phi_A). \quad (8.4)$$

In contrast to Eq. (8.2), this distance measure involves a nonlinear function (the cosine) and thus it can no longer be written in terms of a bilinear form with a metric tensor. The same turns out to be true for curved manifolds: generally it is impossible to express metric distances bilinearly in terms of coordinate differences.

However, the good news is that the concept of a bilinear distance measure still works on the infinitesimal level. In fact, if the two points $A$ and $B$ are infinitesimally close to each other, i.e.,

$$r_A = r, \quad r_B = r + dr, \quad \phi_A = \phi, \quad \phi_B = \phi + d\phi,$$

and if we expand Eq. (8.4) to second order in $dr$ and $d\phi$, we do get a bilinear relation of the form

$$ds^2 = dr^2 + r^2 d\phi^2. \quad (8.5)$$

This expression, which is known as the line element in GR and differential geometry. The line element given above can be written in the form

$$ds^2(\tilde{x}) = g_{ij}(\tilde{x}) \, d\tilde{x}^i \, d\tilde{x}^j, \quad (8.6)$$
8.2 Curved space-time

where \( \tilde{x}^1 = r \) and \( \tilde{x}^2 = \phi \) and

\[
g_{ij}(\tilde{x}) = g_{ij}(r) = \left( \begin{array}{c} 1 \\ r^2 \end{array} \right).
\] (8.7)

Note that this metric tensor depends on \( r \), i.e., it explicitly depends on the position in space. It is still diagonal, reflecting that polar coordinates are locally orthogonal. However, as one can see, the position-dependence does not necessarily imply that the actual space is curved. In fact, for polar coordinates the underlying space is the same \( \mathbb{R}^2 \) as in the case of Cartesian coordinates. The position-dependence rather reflects that the gridlines of the coordinate systems are no longer straight. Therefore, a position-dependent metric tensor does not necessarily imply that the underlying space is actually curved.

**Curved space: Spherical coordinates**

The figure below shows a unit sphere. The unit sphere is a totally symmetric two-dimensional manifold with a constant curvature. The term manifold was introduced by Riemann and stands for a space which locally (on infinitesimal distances) looks like a flat \( \mathbb{R}^n \). The sphere is a very nice example of a curved manifold which is accessible to our three-dimensional imagination, and therefore we will refer to this example frequently.

Spheres are usually parameterized by spherical coordinates, namely, the azimuth angle \( \phi \in [0, 2\pi] \) (longitude) and the polar angle \( \theta \in [0, \pi] \) (latitude) starting from the zenith (north pole). They are related to 3D Cartesian coordinates by

\[
\begin{align*}
x &= r \sin \theta \cos \phi \\
y &= r \sin \theta \sin \phi \\
z &= r \cos \theta
\end{align*}
\] (8.8)

with \( r = 1 \) for a unit sphere. We can now compute the infinitesimal line element in the same way as for polar coordinates:

\[
ds^2 = dx^2 + dy^2 + dz^2 = d\theta^2 + \sin^2 \theta \ d\phi^2.
\] (8.9)

Hence if we define \( \tilde{x}^1 = \theta \) and \( \tilde{x}^2 = \phi \), we can again write the line element as a quadratic form

\[
d\tilde{s}^2 = g_{ij}(\theta) \, d\tilde{x}^i \, d\tilde{x}^j
\] (8.10)

with a \( \theta \)-dependent metric tensor

\[
g_{ij}(\theta) = \left( \begin{array}{c} 1 \\ \sin^2 \theta \end{array} \right)
\] (8.11)

**Remark:** Note that the two metric tensors differ only slightly:

\[
\text{polar: } g_{ij}(r) = \left( \begin{array}{c} 1 \\ r^2 \end{array} \right) \quad \text{unit sphere: } g_{ij}(\theta) = \left( \begin{array}{c} 1 \\ \sin^2 \theta \end{array} \right)
\]

In the first case the radial parameter runs from zero to infinity while in the second case the polar angle is restricted to a finite interval. Is it possible to find out that the first metric
describes a flat space while the second one describes the curved surface? In the following we will see that this is indeed possible.

8.2.2. The concept of space-time as a manifold

Smooth manifolds

In Einstein’s General Relativity space-time is modeled as a Riemann manifold. Roughly speaking a Riemann manifold is a space which looks like the one shown in the figure. Its main features are the following:

- A Riemann manifold is smooth, i.e., on small distances it looks like a flat space.
- On infinitesimal distances, where the space appears to be flat, we still have a scalar product which allows us to compute distances, angles, volumina etc.

In SR we modeled space-time as a flat vector space, representing positions by 4-vectors. In GR we need an entirely different concept. In GR space-time is curved. And even worse, unlike the curved 2D space sketched in the figure above, which is embedded in a surrounding 3D vector space, the curved 3+1-dimensional space-time of GR is not embedded into a higher-dimensional flat vector space in which we could parameterize it by vector components fulfilling certain constraints.

Therefore we have to describe our curved space-time without the help of an embedding vector space. This means that it is no longer possible to represent positions (events) by vectors because there is no concept of curved vectors and curved vector spaces. All what we can say is that space-time consists of points $p$. A point marks a certain position on the manifold.

**In GR space-time is modeled as a smooth Riemann manifold $\mathcal{M}$. Positions are no longer 4-vectors but abstract points $p \in \mathcal{M}$.**

---

2For example, a 2D sphere can be embedded into the 3D flat vector space, where it can be parameterized by coordinates $x, y, z$ constrained by $x^2 + y^2 + z^2 = r^2$. In GR, there is no such obvious embedding, that is, the 3+1-dimensional space-time cannot be embedded naturally in a 5-dimensional vector space.
8.2 Curved space-time

Coordinate systems and maps

Although positions in space-time are no longer represented by 4-vectors but rather by abstract points \( p \in M \), we can still use the concept of *coordinates*, just in the same way as we use coordinates \( \theta, \phi \) on the sphere. A coordinate system is a set of (differentiable) functions which maps a part of the manifold \( U \subseteq M \) uniquely to a set of real numbers. For example, the Earth’s surface can be mapped uniquely to the longitude \( \phi \) and the latitude \( \theta \) (the only exception being the north and the south pole, where the longitude is undefined). These numbers can then be used as coordinates in a flat space. This is essentially what is being done when the Earth’s surface is projected onto a map (see Fig. 8.3).

Interestingly, there is a theorem saying that it is impossible to represent the unit sphere by a single map: at least one point is always missing. But of course we can represent the entire surface by two maps. In mathematics such a collection of maps, which covers the entire manifold, is denoted as an atlas.

Note that a map, although being part of \( \mathbb{R}^n \), does not restore the concept of space-time modeled as a vector space, simply because the vector space axioms are no longer fulfilled on the manifold. In fact, positions and distances between different positions on the manifold can no longer be described in terms of vectors and differences between vectors on the map.

However, *infinitesimal* distances still make sense. That is because in a given coordinate system infinitesimally neighboring points are characterized by infinitesimally differing coordinates. This allows us to define a line element by

\[
 ds^2 = g_{\mu\nu}(x) \, dx^\mu \, dx^\nu \tag{8.12}
\]

with a (not necessarily diagonal) metric tensor \( g_{\mu\nu}(x) \) which depends on the point \( p \) on the manifold or equivalently the coordinates \( x = \{x^\mu\} \). The tensor field \( g_{\mu\nu}(x) \) fully characterizes the geometric properties of the manifold.

8.2.3. Geodesic lines

Imagine that you are driving with your car on a curved manifold. Keep the steering wheel in neutral position and just drive straight ahead. Then your car will follow a special trajectory which in differential geometry is known as a *geodesic line*.

Following a geodesic lines means to “go straight”.

On a 2D sphere the geodesic lines are just the *great circles*. For example, the equator is a great circle, but also if you move north or south at constant longitude, you are moving on a great circle. The great circles are the *geodesics* of the sphere.

Airplanes prefer to cruise along sections of great circles because these are the shortest connections between two points. Plotted on a world map, however, these connections are not necessarily straight. For example, if you fly from Frankfurt to Tokyo, you first see Sweden on your left and later parts of northern Siberia, although on the map this
curved trajectory (green) appears to be longer than a straight line (red). However, in reality the green trajectory is in fact the shortest connection between Frankfurt and Tokyo.

Although there are a couple of subtleties and pitfalls, it turns out that in most cases the geodesic line between two points is indeed the shortest connection. This makes a lot of sense because a geodesic line corresponds to a straight connection in reality (although it may appear curved on the map). Therefore, in order to derive a differential equation for geodesic lines on the map, let us search for the shortest connection by using the good old Lagrange formalism.

**Searching for the shortest connection between two points**

Suppose we want to find the shortest connection between two given points $A$ and $B$ on a curved manifold. The total length $\ell$ of the connection is given by

$$\ell = \int_{c} ds,$$  \hspace{1cm} (8.13)

where $c$ is a curve from $A$ to $B$ and $ds$ is the line element. Let us introduce a suitable coordinate system $p \mapsto x^\mu$ and suppose that the line element in these coordinates is given by

$$ds^2 = g_{\mu\nu}(x) \, dx^\mu \, dx^\nu,$$ \hspace{1cm} (8.14)

where $g_{\mu\nu}(x)$ is the (position-dependent and not necessarily diagonal) metric tensor of the manifold represented in the chosen coordinate system. Moreover, let us parameterize the curve $c$ by a parameter $\lambda$ running from $\lambda_A$ to $\lambda_B$. In this parameterization the line element can be written as

$$ds = \sqrt{g_{\mu\nu}(x) \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}} \, d\lambda^2 = \sqrt{|g_{\mu\nu}(x)\dot{x}^\mu\dot{x}^\nu|} \, d\lambda,$$ \hspace{1cm} (8.15)

allowing us to express the total length of the trajectory as an integral over $\lambda$:

$$\ell = \int_{\lambda_A}^{\lambda_B} \sqrt{|g_{\mu\nu}(x)\dot{x}^\mu\dot{x}^\nu|} \, d\lambda.$$ \hspace{1cm} (8.16)

To find the shortest connection between the two points, we are looking for a curve for which the variation of the total length vanishes, i.e., $\delta \ell = 0$. To this end we recast the problem as mechanical one with the Lagrange function

$$L(x, \dot{x}) = \sqrt{|g_{\mu\nu}(x)\dot{x}^\mu\dot{x}^\nu|}.$$ \hspace{1cm} (8.17)
8.2 Curved space-time

The aim is to find a curve for which
\[ δ\ell = δ \int_{\lambda_1}^{\lambda_2} L(x, \dot{x}) \, d\lambda = 0. \tag{8.18} \]
Applying the machinery of the Lagrange formalism, we can conclude that the trajectory is a solution of the Lagrange equations
\[ \frac{d}{d\lambda} \frac{\partial L}{\partial \dot{x}^\mu} - \frac{\partial L}{\partial x^\mu} = 0. \tag{8.19} \]
It should be noted that in the theory of General Relativity the metric \( g \) depends on the location of the manifold, and that this specific dependence \( g(x) \) encodes the gravitational field. For this reason, both terms in the equations of motion contribute. Taking this position-dependence into account, one can show with some patience that Eq. (8.19) is given by the following differential equation:
\[ \dot{x}^\alpha + \frac{1}{2} \dot{x}^\beta (g_{\beta\mu,\nu} + g_{\beta\nu,\mu} - g_{\mu\nu,\beta}) x^\mu x^\nu = 0, \tag{8.20} \]
where the subscript \( \nu \) with the comma denotes partial derivation with respect to \(\nu\).

**Proof:** Let us restrict ourselves to time-like curves with \( ds \geq 0 \) so that we can omit the modules operation, i.e., \( L = \sqrt{g_{\mu\nu}(x)\dot{x}^\mu\dot{x}^\nu} \). At first we compute
\[ \frac{\partial L}{\partial \dot{x}^\mu} = \frac{1}{2L} \frac{\partial g_{\mu\nu}}{\partial x^\nu} \dot{x}^\nu, \quad \frac{\partial L}{\partial x^\lambda} = \frac{1}{2L} (g_{\rho\nu,\mu} + g_{\rho\mu,\nu}) = \frac{1}{L} g_{\rho\nu} \dot{x}^\nu. \]
From the second term, we have to find the total derivative of \( \lambda \), which we express by using the chain rule as
\[ \frac{d}{d\lambda} \left[ \frac{\partial L}{\partial \dot{x}^\mu} \right] = \frac{\partial}{\partial \dot{x}^\tau} \frac{\partial L}{\partial \dot{x}^\mu} \frac{d\dot{x}^\tau}{d\lambda} + \frac{\partial}{\partial \dot{x}^\mu} \frac{\partial L}{\partial \dot{x}^\rho} \frac{d\dot{x}^\rho}{d\lambda} = \frac{\partial^2 L}{\partial \dot{x}^\tau \partial \dot{x}^\rho} \dot{x}^\tau + \frac{\partial^2 L}{\partial \dot{x}^\mu \partial \dot{x}^\rho} \dot{x}^\rho. \]
The two summands contain the derivatives
\[ \frac{\partial^2 L}{\partial \dot{x}^\tau \partial \dot{x}^\rho} = -\frac{1}{2L^3} \frac{\partial g_{\mu\nu}}{\partial x^\rho} \dot{x}^\mu \dot{x}^\nu g_{\rho\mu,\nu} \dot{x}^\mu + \frac{1}{L} \frac{\partial g_{\rho\nu}}{\partial x^\nu} \dot{x}^\nu, \]
\[ \frac{\partial^2 L}{\partial \dot{x}^\mu \partial \dot{x}^\rho} = \frac{1}{L^2} g_{\rho\nu} \dot{x}^\nu g_{\rho\mu} + \frac{1}{L} g_{\rho\tau}, \]
so that the Lagrange equations, multiplied on both sides by \( 2L \), are given by:
\[ \frac{\partial g_{\mu\nu}}{\partial x^\rho} \dot{x}^\mu \dot{x}^\nu = -\frac{1}{L^2} \frac{\partial g_{\mu\nu}}{\partial x^\tau} \dot{x}^\mu \dot{x}^\tau g_{\nu\rho} \dot{x}^\rho + 2 \frac{\partial g_{\mu\rho}}{\partial x^\nu} \dot{x}^\nu \dot{x}^\tau \dot{x}^\rho - \frac{2}{L^2} g_{\rho\nu} \dot{x}^\nu g_{\rho\mu} \dot{x}^\rho + 2 g_{\rho\mu} \dot{x}^\rho. \]
These rather complicated equations describe the shortest path for any parameterization of the curve. It is now possible to choose a special parameterization so that the equations become simple (similar to choosing a special gauge for the wave equation in electrodynamics). We want to choose the parameterization in such a way that we move along the curve at a constant velocity, i.e., \( ds/d\lambda = \text{const} \) which means that \( L = \text{const} \) (in SR this corresponds to the Eigenzeit parameterization). It should be noted that we are allowed to choose a particular gauge only after the variational calculation has been completed! This causes the first and the third term on the right side to vanish in the above equation. With this gauge the simplified equations read
\[ g_{\rho\nu} \dot{x}^\rho + \frac{1}{2} \left( \frac{\partial g_{\mu\rho}}{\partial x^\nu} \dot{x}^\nu \dot{x}^\rho - \frac{\partial g_{\mu\nu}}{\partial x^\rho} \dot{x}^\rho \dot{x}^\nu \right) = 0 \]
or equivalently
\[ \dot{x}^\alpha + \frac{1}{2} \dot{x}^\beta \left( 2g_{\beta\rho,\nu} - g_{\mu\nu,\beta} \right) x^\mu x^\nu = 0 \]
that can be brought into the desired form. Please keep in mind that the geodesic differential
equations generate geodesic lines with a special parameterization designed to move along
the curve at constant velocity.

In a compact notation, the main result is that a geodesic line, which is the shortest
connection between two given points, is determined by the partial second-order differ-
ential equation

$$\ddot{x}^\alpha = -\Gamma^\alpha_{\mu\nu} \dot{x}^\mu \dot{x}^\nu,$$  \hspace{1cm} (8.21)

where the dot denotes differentiation with respect to the actual line element (Eigenzeit parameterization). The coefficients

$$\Gamma^\alpha_{\mu\nu} = \frac{1}{2} g^{\alpha\beta} (g_{\beta\mu,\nu} + g_{\beta\nu,\mu} - g_{\mu\nu,\beta}),$$  \hspace{1cm} (8.22)

are the so-called the Christoffel symbols. For a given metric tensor it is straightforward
to calculate the Christoffel symbols by computing the partial derivatives given above.
What is the physical meaning of the Christoffel symbols? As we will see below, they are
just the connection coefficients, comparable with $A^\mu$ in the case of electromagnetism.

**First example: Straight lines in $\mathbb{R}^2$ in polar coordinates**

As we have seen before, a two-dimensional plane can be represented by polar coordinates $(x^1, x^2) = (r, \phi)$. In this representation the metric tensor is given by

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}, \quad g^{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & r^{-2} \end{pmatrix}.$$  \hspace{1cm} (8.23)

In this case it is clear that the only non-vanishing partial derivative of the tensor com-
ponents is $g_{22,1}$. Thus the only non-vanishing Christoffel symbols are

$$\Gamma^1_{22} = \Gamma^1_{21} = \frac{1}{2} g^{11} (g_{12,2} + g_{12,2} - g_{22,1}) = \frac{1}{2} g^{11} g_{22,1} = -r,$$

$$\Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{2} g^{22} (g_{21,2} + g_{22,1} - g_{12,2}) = \frac{1}{2} g^{22} g_{22,1} = \frac{1}{r}.$$  \hspace{1cm} (8.24, 8.25)

In this case the equations for the geodesic lines read

$$\ddot{x}^1 + \Gamma^1_{22} \dot{x}^2 \dot{x}^2 = \ddot{r} - r \dot{\phi}^2 = 0,$$

$$\ddot{x}^2 + 2 \Gamma^2_{12} \dot{x}^1 \dot{x}^2 = \ddot{\phi} + \frac{2}{r} \dot{r} \dot{\phi} = 0.$$  \hspace{1cm} (8.26, 8.27)

Amusingly, we have obtained two rather complicated differential equations, just to
describe a straight line in $\mathbb{R}^2$. This is probably one of the worst choices of coordinates
to describe a straight line.

**Second example: Surface of a unit sphere $S^2$**

The surface of a sphere $S^2 \in \mathbb{R}^3$ can be parameterized by

spherical coordinates by two angles $(x^1, x^2) = (\theta, \phi)$, where
the angle $\theta$ is measured starting from the $z$ axis, i.e., from the north pole. The metric tensor reads
\[ g_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2 \theta \end{pmatrix}, \quad g^{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & \sin^{-2} \theta \end{pmatrix}. \] (8.28)

Again the only non-vanishing partial derivative of the tensor components is $g^{22,1}$. Consequently the non-vanishing Christoffel symbols are the following:
\[ \Gamma_2^{12} = \Gamma_2^{21} = \frac{1}{2} g^{22} \left( g_{21,2} + g_{22,1} - g_{12,2} \right) = \frac{1}{2} g^{22} g_{22,1} = \cot \theta. \] (8.30)

The equations for a geodesic line are in this case
\[ \ddot{x}^t + \Gamma^1_{22} \dot{x}^2 \dot{x}^2 = \ddot{\theta} - \dot{\phi}^2 \sin \theta \cos \theta = 0 \] (8.31)
\[ \ddot{x}^2 + 2\Gamma^2_{12} \dot{x}^1 \dot{x}^2 = \ddot{\phi} - 2\dot{\phi} \dot{\theta} \cot \theta = 0 \] (8.32)

They describe large circles on the sphere, which, however, can be ‘tilted’ relative to the equatorial plane and are therefore may come with an oscillating $\theta$ component.

8.2.4. Parallel transport

**Tangent spaces**

Although vectors can no longer be used to describe positions in space-time, vectors are still useful to specify *directions* in space-time. For example, forces can still be added like vectors, the same applies to momenta, velocities, and other directed quantities.

In order to describe directed quantities as vectors, we need a mathematical structure in which these vectors can live. This mathematical structure is called a *tangent bundle*. The idea is to attach to each point $p \in M$ of the manifold (of which there are infinitely many) an individual vector space, the so-called *tangent space* $T_pM$.

Attaching spaces to the points of the manifold is already familiar to us. Remember electrodynamics! Here we attached to each point of space-time a circle with a $U(1)$ symmetry. Now we are doing something similar: we attach to each point of the manifold a flat vector space of the same dimension which is like a Minkowski space, hosting all directed quantities such as velocities and momenta. This means that the $U(1)$ symmetry of electrodynamics is now replaced by the Poincaré symmetry $SO(3,1)$. But unlike electrodynamics, where the $U(1)$ circle represents an *intrinsic* degree of freedom, the tangent space now represents directions in space-time itself, i.e., it refers to the *extrinsic* degrees of freedom.
**Coordinate basis**

In order to represent tangent vectors in the tangent space we have to define a suitable basis. In principle we are free to choose any basis in each of the attached tangent spaces. However, it is useful to define a special basis along the mesh of the coordinate system. This basis is called the *coordinate basis* of the tangent space.

An example of a tangent space with a coordinate basis is shown in Fig. 8.4. The left panel shows the flat $\mathbb{R}^2$ with polar coordinates. At a certain position, labeled by the coordinates $(\phi, r)$ and indicated as a red bullet the figure, we attach a tangent space which can be seen as approximating in the immediate neighborhood of the red bullet. In this tangent space we choose two vectors which basically follow the mesh lines of the coordinate system. More specifically, $\vec{e}_\phi$ points in the direction where the coordinate $\phi$ varies while the other coordinate $r$ is kept fixed. Conversely, $\vec{e}_r$ points in the direction where the coordinate $r$ varies, keeping the angle $\phi$ fixed. In the corresponding map, shown in the right panel, this basis looks like a standard basis.

More generally, $\vec{e}_\mu$ is a vector in the direction where the coordinate $x^\mu$ increases while all other coordinates are kept fixed. Notice that this so-called *coordinate basis* is not necessarily orthonormal with respect to the given metric.

**Remark:** Really cool experts do not use the notation $\vec{e}_\mu$, instead they would use partial derivatives $\partial_\mu$ as a standard notation for basis vectors. For beginners this seems to be strange since a derivative is an operator rather than vector. However, a partial derivative does exactly what we outlined above, namely, it looks for a variation of some function of $x_\mu$ while keeping all other coordinates constant.

**Transport of objects**

In order to understand how differential geometry works, it is useful to think of the transport of objects, in particular of scalars and vectors. Imagine a captain on a ship carrying these objects from one place to the other.

- **Transport of scalars:**
  The transport of scalars is very simple. A scalar is basically a number, e.g. 2.718. The captain takes the scalar on board and leaves it unchanged during the travel. At the destination the scalar still has same value 2.718.

- **Transport of vectors:**
The transport of vectors is much more demanding (see Fig. 8.5). The only place where vectors can live in is the local tangent space attached to the current position. On departure, the captain receives a vector in the local tangent space. It points at a certain direction, for example, to the left relative to the ship, as shown in the figure. What happens to this direction during the travel? This is a highly nontrivial question, since as the ship moves, the attached tangent spaces change continuously. Therefore we need a mechanism to move a vector from one tangent space to the next.

Suppose that the captain keeps the rudder in neutral position so that the ship sails straight along a geodesic line. Then it is reasonable that the vector should be transported in such a way that its angle relative to the ship doesn’t change. On a map, however, it is possible that the same trajectory appears to be curved, and on the map the velocity vector seems to change its direction permanently (green vector in Fig. 8.5).

In order to compute how the vector changes on the map during parallel transport, let us consider the transport over an infinitesimal distance from a point \( p \) to another point \( p' \), as shown in Fig. 8.6. Here we expect that the transported vector \( v \) will change only by a small amount \( \delta v \). Moreover, it is reasonable to expect that this change will depend linearly on the displacement \( \delta x \) and on the vector itself. In other words, the change on the map is expected to be given by a bilinear function \( \Gamma \) as

\[
\delta v = -\Gamma_p(v, \delta x),
\]

where the minus sign is just a convention. The function \( \Gamma \) may depend on the current position \( p \in \mathcal{M} \), as expressed by the subscript. Note that this formula is not yet correct because shortly before we argued that points on the manifold can no longer be described by vectors, and therefore a finite displacement cannot be described by a vector. However, if the displacement is infinitesimal and if we divide by the corresponding
change of the parameter $\delta \lambda$ along the geodesic line, we can rewrite this expression as

\[
\dot{v} = -\Gamma^p_{\mu \nu}(v, \dot{x}). \tag{8.34}
\]

Now this equation makes sense: $\dot{x}$ is the tangent vector along the trajectory (the velocity vector) which lives in tangent space. Likewise the transported vector $v$ lives in tangent space. The equation above tells us how the transported vector represented on the map changes with $\lambda$ if we move in a certain direction with a certain velocity given by the vector $\dot{x}$. Assuming this relation to be bilinear, we can write it in components as

\[
\dot{v}^\alpha = -\Gamma^\alpha_{\mu \nu}(p) v^\mu \dot{x}^\nu. \tag{8.35}
\]

It turns out that the coefficients $\Gamma^\alpha_{\mu \nu}$ are just the Christoffel symbols introduced above (not proven here). In particular, if the transported vector is the velocity of the ship itself, i.e., $v = \dot{x}$, then the equation given above reduces to the geodesic equation (8.21)

\[
\ddot{x}^\alpha = -\Gamma^\alpha_{\mu \nu}(p) \ddot{x}^\mu \dot{x}^\nu. \tag{8.36}
\]

**Covariant derivative**

As in electrodynamics, the covariant derivative is the ‘true’ generator of translations, that is, it tells us how a quantity changes as we move along the manifold. Again it depends on the type of the quantity to which the derivative is applied:

- If the quantity is a scalar field $f(p)$, for example the air pressure or the temperature, the change that the sailor will experience while traveling is just given by the gradient of the field contracted with the velocity:

  \[
  \dot{f} = (\partial_{\mu}f) \dot{x}^\mu.
  \]

  This is exactly the same formula that we would also have in a flat space.

- If the quantity is a vector field $v(p)$ in the tangent bundle, for example the wind velocity that the sailor experiences while traveling, the true change of the vector has to be measured relative to a parallel-transported one. This boils down to

  \[
  \dot{v}^\alpha = \left(\partial_{\mu}v^\alpha + \Gamma^\alpha_{\mu \nu} v^\nu\right) \dot{x}^\mu
  \]

  or short

  \[
  \dot{v} = (D_{\mu}v) \dot{x}^\mu.
  \]

Recall electrodynamics: here the covariant derivative was defined as $D_{\mu} = \partial_{\mu} - iA_{\mu}$. We interpreted this covariant derivative in the following way: as a generator of translations, we have to do two things. On the one hand, we have to translate the object by an ordinary derivative. On the other hand, we have to account for gauge transformations or real twists on the $U(1)$ circle. These corrections are generated by the imaginary unit (which is the generator of $U(1)$ transformations on the complex circle) and the amount
of the correction which is controlled by the field \( A_\mu \).

In General Relativity the situation is completely analogous. Also here the covariant derivative is considered as the generator of translations. In order to translate an object we have to do two things: on the one hand we have to generate the bare translation in space, as it is done by applying the ordinary partial derivative. In addition, we have to account for possible "gauge transformations" (varying choices of the coordinate system in neighboring tangent spaces) as well as real twists of space-time (an underlying curvature, i.e., gravity). These additional corrections are accounted for by the Christoffel symbols (also called the connection in differential geometry).

As in electrodynamics, where \( iA_\mu \) generates \( U(1) \)-transformations between infinitesimally neighboring 1D circles, the Christoffel symbols \( \Gamma^\alpha_{\mu\nu} \) perform transformations between neighboring 4D tangent spaces.

### 8.2.5. Curvature

How can we detect whether manifold given in terms of its metric tensor and the corresponding Christoffel symbols is really curved (and not just apparently as in the case of polar coordinates)? More specifically, how can we detect curvature if we cannot access a surrounding flat embedding space?

Remember how we detected real physical twists of the \( U(1) \)-tubes in electrodynamics. To detect a factual twist (rather than a gauge artifact) we had to move along the closed contour, and a mismatch in the internal coordinate \( \phi \) indicated the existence of a real twist. Since there are six linearly independent possibilities to orient the closed contour in a 3+1-dimensional space, we ended up with six field components, three spatial-temporal ones (electric field) and three spatial ones (magnetic field).

We now repeat this line of thinking. Imagine a sailor transporting a vector on a closed contour sketched in the figure. The sailor has to keep the angle of the vector relative to the ship as long as he is moving on a section of the geodesic line. If he turns to the left by 90°, as it happens three times along the trajectory, the transported vector should not change its orientation. If the equatorial section of the contour covers one quarter of the circumference, the figure illustrates that the sailor will return to the origin with the transported vector pointing into different direction. This mismatch indicates that the underlying space enclosed by the countour is curved.

In the case of electrodynamics, we considered an infinitesimally small contour and Taylor-expanded the contributions along the edges to lowest order. This led us to the field strength tensor \( F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha \). In General Relativity we can do a completely analogous calculation, leading us to the so-called Riemann curvature tensor

\[
R^\mu_{\nu\alpha\beta} = \partial_\alpha \Gamma^\mu_{\nu\beta} - \partial_\beta \Gamma^\mu_{\nu\alpha} + \Gamma^\rho_{\nu\beta} \Gamma^\mu_{\rho\alpha} - \Gamma^\rho_{\nu\alpha} \Gamma^\mu_{\rho\beta} \quad (8.37)
\]
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While in electrodynamics the internal degree of freedom is one-dimensional and therefore does not require additional indices, the Christoffel symbols established the connection of infinitesimally neighbored tangent spaces and therefore they involve more indices. This is the reason why the curvature tensor $G$ involves not only two but four indices.

Notice that also the formula is somewhat different. The first two terms on the right hand side clearly resemble the antisymmetric expression that we know from electrodynamics. The other two terms, however, are new to us. They are quadratic in the Christoffel symbols and reflect the circumstance that the transformations, that bring us from one tangent space to the other (including rotations and Lorentz boosts) do not commute. In other words, the group $U(1)$, the gauge group of electrodynamics, is commutative while the group $SO^+(3, 1)$, which is the gauge group of the tangent spaces, is non-commutative. The non-commutativity leads to the two additional non-linear terms in the curvature tensor.

**Einstein field equations**

Finally let us briefly discuss the second pillar of General Relativity, namely, Einstein’s field equations. These equations tell us that the local curvature is basically proportional to the energy-matter content at this point. In the beginning of the lecture we have seen that the energy-matter content is encoded in the energy momentum tensor $T_{\mu\nu}$. However, this tensor has only two and not four indices. Therefore, we have to look for meaningful contractions of the curvature tensor. It turns out that the only meaningful contraction is the so-called Ricci tensor

\[ R_{\mu\nu} := R^\rho_{\mu\rho\nu}. \]  

which can be contracted further to a curvature scalar

\[ R = R^\mu_{\mu}. \]  

(8.38)

(8.39)

To keep the story short, it took Einstein almost 10 years to find the following correct relation between curvature and energy-momentum tensor:

\[
R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}
\]  

(8.40)

Here $G$ is just Newton’s gravitational constant. The coefficient $\Lambda$, known as the cosmological constant was already introduced by Einstein in order to stabilize a static universe. This idea was abandoned when it became clear that such a Universe would not be stable against perturbations and that a Big Bang scenario is more consistent with our observations. Surprisingly, the cosmological constant had an amazing comeback some 20 years ago, when it was discovered that the Universe expands more quickly than expected. This accelerated expansion was attributed to a mysterious kind of anti-gravity, known as dark energy (not to be confused with dark matter which has an attractive interaction and therefore tends to clump). Until today, the physical nature of the dark...
energy, whose existence is undisputed, is not yet understood.

There are so many exciting unsolved problems around! My personal impression is that Physics became much more interesting in recent decades.

**Summary 8.2.1.** The most important formulas of General Relativity:

\[
\Gamma^\alpha_{\mu\nu} = \frac{1}{2} g^\alpha_{\beta\mu} \left( g_{\beta\nu,\mu} - g_{\mu,\beta} \right)
\]

\[
\ddot{x}^\alpha + \Gamma^\alpha_{\mu\nu} \dot{x}^\mu \dot{x}^\nu = 0
\]

\[
R^\mu_{\nu\alpha\beta} = \partial_{\alpha} \Gamma^\mu_{\nu\beta} - \partial_{\beta} \Gamma^\mu_{\nu\alpha} + \Gamma^\rho_{\nu\beta} \Gamma^\mu_{\rho\alpha} - \Gamma^\rho_{\nu\alpha} \Gamma^\mu_{\rho\beta}
\]

\[
R_{\mu\nu} = R^\rho_{\mu\rho\nu} \quad R = R^\mu_{\mu}
\]

\[
R_{\mu\nu} = \frac{1}{2} \left( 8\pi G \frac{T_{\mu\nu}}{c^4} + \Lambda g_{\mu\nu} \right)
\]  

**Alternative form of the field equations**

If one contracts both sides of the field equations with \( g_{\mu\nu} \) one obtains a scalar relation

\[
-R + 4\Lambda = \frac{8\pi G}{c^4} T,
\]  

where \( R = R^\mu_{\mu} \) is the curvature scalar and \( T = T^\mu_{\mu} \) is the trace over the energy-momentum tensor. This scalar relation can be used to bring the second term in the field equations to the other side by expressing \( R \) in \( G \) and \( T \). This leads us to an alternative and fully equivalent form of the field equations:

\[
R_{\mu\nu} = \Lambda g_{\mu\nu} + \frac{8\pi G}{c^4} \left( T_{\mu\nu} - \frac{1}{2} T g_{\mu\nu} \right).
\]  

The alternative form of Einstein’s field equations shows that for a given distribution of matter and energy we can always calculate the corresponding Ricci tensor. However, this does not yet fully determine Riemann’s curvature tensor (the one with four indices), and so the question arises whether this rank-4 tensor contains more information than the rank-2 Ricci tensor, and if so, which kind of information.

The field equations already give us a physical hint: in vacuum, with the cosmological constant set to zero, we have \( R_{\mu\nu} = 0 \). But this does not imply that that space-time is entirely flat in the sense that \( R^\mu_{\nu\alpha\beta} = 0 \). For example, one can show that the remaining freedom allows for gravitational waves.

**The gauge group of GR**

Although so far the presentation of GR was sketched only superficially, we can already see that GR in electrodynamics are constructed in the same spirit. There are many elements in both theories which are similar:
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<table>
<thead>
<tr>
<th>Electrodynamics</th>
<th>General Relativity</th>
</tr>
</thead>
<tbody>
<tr>
<td>1D intrinsic circle</td>
<td>4D tangent space</td>
</tr>
<tr>
<td>$A_\mu$</td>
<td>$\Gamma^\alpha_{\mu\beta}$</td>
</tr>
<tr>
<td>$F_{\mu\nu}$</td>
<td>$R^a_{\mu\nu\beta}$</td>
</tr>
</tbody>
</table>

The essential difference is that the tangent space addresses *extrinsic* degrees of freedom (space-time itself) while the $U(1)$-circle of electrodynamics is understood as an *intrinsic* (and possibly even compactified) degree of freedom. Nevertheless we expect both theories to be constructed in the same conceptual way. As in electrodynamics, where we are free to change the coordinate systems in each of the circles, thereby changing $A_\mu$ but not the physical fields $F_{\mu\nu}$, it should be possible in GR to vary the coordinate system of the tangent spaces, thereby changing the Christoffel symbols while keeping the physical fields in the curvature tensor invariant. What is the gauge group of GR?

The easy guess would be that the gauge transformations are just the Poincaré transformations $SO^+(3,1)$. But this is only partly true. In fact, in the beginning of this course we learned that Poincaré group has the special property of leaving the metric tensor invariant. In fact, there is much more freedom in GR, namely, we can choose any coordinate system on the manifold which is defined in reasonably differential coordinate functions. As we will see below, this includes accelerated (non-inertial) frames. Therefore, the gauge group of GR is really huge, namely, it includes all mappings between all kinds of inertial and non-inertial coordinate systems that do not affect the topology and the differentiability, or in other words, that do not tear space-time apart. In differential geometry, such maps are known as *diffeomorphisms*.

8.3. Accelerated frames

To get a feeling what gravity and any other form of acceleration is like, we here consider the properties of accelerated frames within the framework of special relativity.

8.3.1. Constant acceleration

Let us imagine a rocket with constant acceleration $\vec{a}$. What will be its trajectory in a flat Minkowski space? How does the world look like from the perspective of the rocket?

In Special Relativity, the trajectory of the spacecraft in space-time is described by a 4-vector $x(\lambda)$, where $\lambda$ is some parameter for which we shall choose the proper time $\lambda = \tau$ (Eigenzeit). This allows us to define a 4-velocity and a 4-acceleration (cf. Sect. 3.1.2 on page 53) by

$$u = \begin{pmatrix} \gamma c \\ \gamma \vec{v} \end{pmatrix}, \quad a = \begin{pmatrix} \gamma \gamma c \\ \gamma^2 \vec{a} + \gamma \gamma \vec{v} \end{pmatrix},$$

where $\vec{v} = \frac{d}{d\tau} \vec{x}$ is the usual 3-velocity. Now suppose that we are observing the rocket
from an inertial frame which is co-moving in the moment of observation, that is, we have \( \vec{v} = 0 \) in the moment of observation. In this co-moving (cm) frame the 4-velocity and the 4-acceleration are given by

\[
\mathbf{u}_{cm} = \begin{pmatrix} c \\ 0 \end{pmatrix}, \quad \mathbf{a}_{cm} = \begin{pmatrix} 0 \\ \gamma \vec{a} \end{pmatrix}.
\]

(8.44)

Let us now assume that the rocket flies in \( x \)-direction. Then we can apply a Lorentz boost in \( x \) direction to express the 4-acceleration in the rest frame of the observer on Earth:

\[
\mathbf{a} = \begin{pmatrix} \gamma \gamma v_x/c \\ \gamma v_x/c \gamma \\ 1 \end{pmatrix} \begin{pmatrix} 0 \\ \alpha_x \\ \alpha_y \\ \alpha_z \end{pmatrix} = \begin{pmatrix} \gamma \alpha_x v_x/c \\ \gamma \alpha_x \\ \gamma \alpha_y \\ \gamma \alpha_z \end{pmatrix}.
\]

(8.45)

This has to be compared with Eq. (8.43):

\[
\begin{pmatrix} \gamma \alpha_x v_x/c \\ \gamma \alpha_x \\ \alpha_y \\ \alpha_z \end{pmatrix} = \begin{pmatrix} \gamma \dot{\gamma} c \\ \gamma^2 a_x + \gamma \dot{\gamma} v_x \\ \gamma^2 a_y + \gamma \dot{\gamma} v_y \\ \gamma^2 a_z + \gamma \dot{\gamma} v_z \end{pmatrix}.
\]

(8.46)

The first component of this identity gives

\[
\gamma \dot{\gamma} = \gamma \alpha_x \frac{v_x}{c^2}.
\]

(8.47)

Inserting this relation into the second component yields

\[
\gamma \alpha_x = \gamma^2 a_x + \gamma \dot{\gamma} v_x = \gamma^2 a_x + \gamma \alpha_x \frac{v_x^2}{c^2} \Rightarrow \gamma^2 a_x = \gamma \alpha_x (1 - \frac{v_x^2}{c^2}) = \frac{1}{\gamma} \alpha
\]

(8.48)

\[
\Rightarrow \alpha_x = \gamma^3 a_x
\]

while the other two components reduce to

\[
\alpha_y = \gamma^2 a_y, \quad \alpha_z = \gamma^2 a_z.
\]

(8.49)

Now let us assume that the rocket is accelerated only in \( x \)-direction with \( \alpha_x \equiv \alpha \). Then, seen from the frame of an observer on earth, its trajectory is described by the differential equation

\[
\frac{d^2}{dt^2} x(t) = \frac{\alpha}{\gamma^3} = (1 - \frac{v_x^2}{c^2})^{3/2} \alpha.
\]

(8.50)

Since the rocket is initially at rest \( v(0) = 0 \), its velocity is found to increase as

\[
v_x(t) = \frac{\alpha t}{\sqrt{1 - \frac{\alpha^2 t^2}{c^2}}}
\]

(8.51)
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and the position as a function of time is given by

\[ x(t) = \sqrt{c^2 t^2 + \frac{c^4}{\alpha^2}} + \text{const} \]  

(8.52)

This can also be written in the hyperbolic form

\[ (x(t) - \text{const})^2 - c^2 t^2 = \frac{c^4}{\alpha^2}. \]  

(8.53)

Remark: Taylor-expanding these results we get:

\[ v_x(t) = \alpha t + \frac{\alpha^3 t^3}{2c^2} + \frac{3\alpha^5 t^5}{8c^4} + \mathcal{O}(t^7) \]

\[ x(t) = \text{const} + \frac{\alpha^2}{c} + \frac{1}{2}\alpha^2 t^2 - \frac{\alpha^4 t^4}{8c^2} + \mathcal{O}(t^6) \]  

(8.54)

The terms highlighted in green color are just the non-relativistic contributions.

At this point it is useful to compute the proper time \( \tau \) elapsed on board of the rocket as a function of \( t \):

\[ \tau(t) = \int_0^t \frac{dt}{\gamma} = \int_0^t \sqrt{1 - \frac{v_x(t)}{c}} \, dt \]

\[ = \frac{c}{\alpha} \ln \left( \frac{\alpha t}{c} + \sqrt{1 + \frac{\alpha^2 t^2}{c^2}} \right) \]  

(8.55)

This relation can be inverted:

\[ t(\tau) = \frac{c}{\alpha} \sinh \left( \frac{\alpha \tau}{c} \right) \]  

(8.56)

implying that

\[ x(\tau) = \frac{c^2}{\alpha} \cosh \left( \frac{\alpha \tau}{c} \right). \]  

(8.57)

Again we can see that the trajectory of the rocket is a hyperbola:

\[ x^2(\tau) - c^2 t^2(\tau) = \frac{c^4}{\alpha^2}. \]  

(8.58)

This makes a lot of sense! Even though the rocket accelerates constantly in its own rest frame, its velocity cannot exceed that of the speed of light in the frame from the observer on Earth. What we learn here is that constant acceleration always corresponds to hyperbolas in Minkowski space.

**Constant acceleration ⇔ Hyperbolas in Minkowski space**

8.3.2. Rindler coordinates
Rindler coordinates \((x_r, t_r)\) are special non-inertial coordinates in which a trajectory with constant coordinate \(x_r\) corresponds to a motion with constant acceleration. Therefore, the gridlines of constant \(x_r\) in Rindler coordinates, when seen from the rest frame, are hyperbolas. Interpreting the acceleration as gravity, Rindler coordinates are often regarded as a free-fall system in a constant gravitational field.

Rindler coordinates are defined by
\[
\begin{align*}
ct &= x_r \sinh(at_r) \\
x &= x_r \cosh(at_r) \\
y &= y_r \\
z &= z_r
\end{align*}
\]  

(8.59)

where \(ct_r, x_r, y_r, z_r\) are the Rindler coordinates while \(ct, x, y, z\) are the usual coordinates in the laboratory inertial frame. This relation can be inverted as follows:
\[
\begin{align*}
t_r &= \frac{1}{a} \arctanh(ct/x) \\
x_r &= \sqrt{x^2 - c^2 t^2} \\
y_r &= y \\
z_r &= z
\end{align*}
\]  

(8.60)

Line element

We can now compute the line element in Rindler coordinates. First we express infinitesimal changes in the ordinary coordinates by
\[
\begin{align*}
dt &= \frac{\partial t}{\partial x_r} \, dx_r + \frac{\partial t}{\partial t_r} \, dt_r = \frac{ax_r}{c} \cosh(at_r) \, dt_r + \frac{1}{c} \sinh(at_r) \, dx_r \\
dx &= \frac{\partial x}{\partial x_r} \, dx_r + \frac{\partial x}{\partial t_r} \, dt_r = \cosh(at_r) \, dx_r + ax_r \sinh(at_r) \, dt_r.
\end{align*}
\]  

(8.61)

This allows us to compute the line element:
\[
ds^2 = dx^2 + dy^2 + dz^2 - c^2 \, dt^2 = dx_r^2 + dy_r^2 + dz_r^2 - a^2 x_r^2 \, dt_r^2.\]  

(8.62)
Consequently the metric tensor is diagonal and depends on $x_r$ in its temporal matrix element:

$$g_{\mu\nu}(x_r) = \begin{pmatrix} -a^2 x_r^2 & 1 \\ 1 & 1 \\
 1 & 1 \end{pmatrix}. \quad (8.63)$$

Thus, in Rindler coordinates, time is basically rescaled according to the position in space while the spatial components remain unaffected.

**Christoffel symbols and curvature**

As we have seen above, for a given metric it is straightforward to compute the corresponding Christoffel symbols and the Riemann curvature tensor. If we do that carefully we find that in 1+1 dimensions the non-vanishing Christoffel symbols are given by

$$\Gamma^0_{01} = \Gamma^0_{10} = \frac{1}{x_r}, \quad \Gamma^1_{00} = \alpha^2 x_r. \quad (8.64)$$

Using this result to compute the Riemann curvature tensor we find that

$$R^a_{b\mu\nu} = 0. \quad (8.65)$$

Thus, the space-time described by Rindler coordinates is flat! This is no surprise because we did not include any sources (masses) in the description and we did not consider nontrivial oscillatory solutions describing gravitational waves.

**Rindler horizon**

Rindler coordinates may be interpreted as infinitely many rockets placed on the positive $x$-axis at the initial positions $x = x_r > 0$. All rockets start at the time $t = t_r = 0$. The idea is that each rocket flies in such a way that the Rindler coordinate $x_r$ remains constant. To this end each rocket has to accelerate uniformly, but with a different acceleration, namely

$$\alpha = \frac{c^2}{x_r}. \quad (8.66)$$

At first glance this seems to be paradoxical: why do uniformly accelerated rockets flying in a row have to accelerate differently in order to maintain a constant distance? This is contradicting our Newtonian everyday experience. But thinking twice it becomes clear that a rocket behind another rocket is red-shifted, therefore it has to push harder to catch up (see Fig. 8.7).

The required acceleration increases as $x_r \to 0$ and finally diverges at the origin. This is the so-called **Rindler horizon** beyond which it is impossible for a uniformly accelerated rocket to catch up, no matter how strongly it accelerates. And indeed, it is even impossible for light to catch up. Consequently, Rindler coordinates are only defined in a certain part of the Minkowski space, namely, in the so-called **Rindler wedge** (see figure).
Unruh effect

The Unruh effect is a controversial conjecture stating that the Rindler horizon emits thermal radiation at some temperature. This phenomenon is analogous to the Hawking radiation of black holes. A thorough understanding of the Unruh effect requires quantum field theory, but nevertheless we can derive it partly by a simple dimensional analysis. To start with, the finite distance of the Rindler horizon induces a certain timescale $\tau$ and thereby a typical frequency $\omega$:

$$\tau_r = \frac{x_r}{c} = \frac{c}{\alpha} \Rightarrow \omega = \frac{2\pi}{\tau_r} = \frac{2\pi\alpha}{c}.$$  

Let us now associate this scale with an energy by simply using the quantum-mechanical relation

$$E = \hbar \omega = \frac{2\pi\hbar\alpha}{c},$$

and since energies can always be translated into a temperature $E \approx k_BT$ we arrive at $T = \frac{2\pi\hbar\alpha}{\hbar c k_B}$. This result differs from the correct result only by a constant factor $4\pi^2$:

$$T_{\text{Unruh}} = T = \frac{\hbar\alpha}{2\pi c k_B T} \quad (8.67)$$

Thus the Rindler horizon radiates at a certain temperature, just in the same way as the horizon of a black hole emits Hawking radiation. However, for accelerations in our everyday world this temperature is so extremely small that we can neglect it.

Example: A formula-1 car accelerates with roughly $\alpha = 1.5g$. As the car accelerates, a Rindler horizon emerges behind the driver at a distance of

$$x_r = \frac{c^2}{\alpha} \approx 6 \cdot 10^{15} \text{m}$$

which is roughly 2/3 of a light year. This timescale – 2/3 of a year – can be translated into a frequency $\omega$ with an associated energy $k_BT = \hbar\omega$. The corresponding temperature is then

$$T = \frac{\hbar\alpha}{2\pi k_B c} \approx 3.81 \cdot 10^{-19} \text{ K}.$$ 

So for a formula-1 driver the Rindler horizon is far away and the associated temperature is negligible small – no reason to worry about!
8.4. Black holes

8.4.1. Schwarzschild solution

We owe to Karl Schwarzschild (1873-1916) the simplest but perhaps most important exact solution of Einstein’s field equations. When World War I broke out in 1914, he volunteered to join the army as many German Jews did during that time. Serving for the army in the war against Russia, he was often in a kind of standby and had still enough time to work on physical problems. During this time he found in 1915 a solution of the Einstein field equation and discovered the radius named after him. He later returned to Germany as an disabled veteran and died in 1916.

The so-called Schwarzschild solutions are mainly based on the assumption of radial symmetry and are therefore suitable for the description of stars, neutron stars and black holes, but they are also the basis for simple cosmological models. Similar to the theory of Newton, in which one considers the shape of the field inside and outside a star separately, there is an interior and an exterior Schwarzschild metric. We will first deal with the exterior Schwarzschild metric.

**Exterior Schwarzschild metric in vacuum**

The exterior Schwarzschild metric is a radially symmetric solution of Einstein’s field equations in vacuum $R_{\mu\nu} = 0$. Starting point is the observation that the flat Minkowski metric $\eta_{\mu\nu}$ can be written in spherical coordinates $t, \tilde{r}, \theta, \phi$ as the line element

$$d s^2 = -c^2 dt^2 + d\tilde{r}^2 + \tilde{r}^2 (d\theta^2 + \sin^2 \theta \, d\phi^2). \quad (8.68)$$

However, if there is a mass in the center that gives the surrounding space a curvature, the flat Minkowski metric is no longer valid. Nevertheless the metric is expected to maintain its spherical symmetry. In order to guess a suitable line element, a possible Ansatz would be to multiply each of the terms in the equation given above by functions $f, g, h$ which depend only on the radius $\tilde{r}$:

$$d s^2 = -f(\tilde{r})c^2 dt^2 + g(\tilde{r}) d\tilde{r}^2 + h(\tilde{r})\tilde{r}^2 (d\theta^2 + \sin^2 \theta \, d\phi^2). \quad (8.69)$$

Only two of the three functions are independent because it is always possible to rescale the radial coordinate by

$$r = \tilde{r}\sqrt{h(\tilde{r})}. \quad (8.70)$$

Thus, setting $c = 1$ a possible ansatz reads

$$d s^2 = -B(r)c^2 dt^2 + A(r) dr^2 + r^2 (d\theta^2 + \sin^2 \theta \, d\phi^2), \quad (8.71)$$

where $A(r)$ and $B(r)$ are positive functions which have to be determined by solving Einstein’s field equations.

---

In order to preserve the signature $- + + +$ of the metric, the function $h$ has to meet certain conditions which are not mentioned here.
Solution of the field equations

In the following it will be convenient to express the two functions in terms of exponential functions $A(r) = e^{\alpha(r)}$ and $B(r) = e^{\beta(r)}$. Using this notation the metric tensor corresponding to the line element given above is diagonal:

$$g_{00} = g_{tt} = -e^\beta, \quad g_{11} = g_{rr} = e^\alpha, \quad g_{22} = g_{\theta\theta} = r^2, \quad g_{33} = g_{\phi\phi} = r^2 \sin^2 \theta$$

(8.72)

The corresponding metric tensor with upper indices $g^{\mu\nu}$, which is the inverse of the one with lower indices, is obtained by taking the reciprocals along the diagonal. Because of the spherical symmetry the second and the third term the definition of the Christoffel symbols cancel. The non-vanishing Christoffel symbols therefore read

$$\Gamma^0_{01} = \Gamma^0_{10} = \frac{1}{2} \beta', \quad \Gamma^1_{00} = \frac{1}{2} \beta' e^{\beta - \alpha}, \quad \Gamma^1_{11} = \frac{1}{2} \alpha',$$

$$\Gamma^1_{22} = -r e^{-\alpha}, \quad \Gamma^1_{33} = -r e^{-\alpha} \sin^2 \theta, \quad \Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{r}, \quad \Gamma^3_{13} = \Gamma^3_{31} = \frac{1}{r}, \quad \Gamma^3_{23} = \Gamma^3_{32} = \frac{1}{\tan \theta}.$$  

(8.73)

Having computed the Christoffel symbols it is straightforward to calculate the matrix elements of the curvature tensor

$$R^0_{101} = -\frac{1}{2} \beta'^{\prime\prime} - \frac{1}{4} \beta'^2 + \frac{1}{4} \alpha' \beta'$$

$$R^0_{202} = -\frac{1}{2} r e^{-\alpha} \beta'$$

$$R^0_{303} = -\frac{1}{2} r e^{-\alpha} \beta' \sin^2 \theta$$

$$R^1_{212} = -\frac{1}{2} r e^{-\alpha} \alpha'$$

$$R^1_{313} = -\frac{1}{2} r e^{-\alpha} \alpha' \sin^2 \theta$$

$$R^2_{323} = (1 - e^{-\alpha}) \sin^2 \theta.$$  

(8.74)

By contraction, the corresponding Ricci-Tensor is given by

$$R_{00} = e^{\beta - \alpha} \left( \frac{1}{2} \beta'^{\prime\prime} + \frac{1}{4} \beta'^2 - \frac{1}{4} \alpha' \beta' + \frac{1}{r} \beta' \right),$$

$$R_{11} = -\frac{1}{2} \beta'^{\prime\prime} - \frac{1}{4} \beta'^2 + \frac{1}{4} \alpha' \beta' + \frac{1}{r} \alpha',$$

$$R_{22} = 1 + e^{-\beta} \left( -\frac{1}{2} r \alpha' + \frac{1}{2} r \beta' - 1 \right),$$

$$R_{33} = R_{22} \sin^2 \theta.$$  

(8.75-8.78)

and, contracting further, we end up with the Ricci scalar

$$R = \frac{e^{-\alpha}}{2r^2} \left( 4(e^\alpha - 1) + r((\alpha' - \beta')(4 + r \beta') - 2r \beta'^{\prime\prime}) \right).$$  

(8.79)
Outside the star or the black hole (in the exterior space) we have a vacuum without masses and consequently $R_{\mu
u} = 0$. From the first two equations we can derive that

$$\alpha' + \beta' = 0 \quad \Rightarrow \quad \alpha + \beta = \text{const.} \quad (8.80)$$

As another constraint, we require that the Schwarzschild metric describes a massive object in an otherwise empty space so that the metric in a large distance from the object is expected to tend towards the flat Minkowski metric, i.e.,

$$\lim_{r \to \infty} \alpha = \lim_{r \to \infty} \beta = 0 \quad \Rightarrow \quad \alpha = -\beta \quad \Rightarrow \quad \text{const} = 0. \quad (8.81)$$

Because of $R_{22} = 0$ we are therefore led to the differential equation

$$1 - e^{-\beta} + r \beta' = 0 \quad (8.82)$$

with the solution

$$e^{\beta(r)} = 1 - \frac{r_s}{r}, \quad (8.83)$$

where $r_s$ is an integration constant. Thus, the exterior Schwarzschild metric is given by

$$ds^2 = -\left(1 - \frac{r_s}{r}\right) dt^2 + \left(1 - \frac{r_s}{r}\right)^{-1} dr^2 + r^2 \left(d\theta^2 + \sin^2 \theta \, d\phi^2\right). \quad (8.84)$$

Remark: One can show that the spherical symmetry determines the form of the metric tensor to a large extent and that one can always find suitable coordinate transformations bringing the metric into the form given above. In the derivation outlined above we have assumed that the metric is time-independent, suggesting that one obtains a static solution. However, this only means that the metric looks static in the chosen coordinates. In fact, one can show that the Schwarzschild metric is even more general and able to describe time-dependent scenarios such as oscillating or uniformly collapsing objects.

The so-called Birkhoff theorem tells us that the exterior gravitational field of an object with spherical mass distribution depends only on the total mass $M$ but not on the specific kind of the distribution. In so far the situation is similar as an Newtonian mechanics. Moreover, one can show that the exterior Schwarzschild metric is the only asymptotically flat solution of that kind.

**Schwarzschild radius**

The integration constant $r_s$ is called *Schwarzschild radius*. In order to determine this constant quantitatively, we consider the Schwarzschild metric at large distance from the object with mass $M$ in the center. Here we can expand the metric to lowest order around the Minkowski metric. Doing so it can be shown that the component

$$g_{00} \approx 1 + h_{00} \approx 1 + \frac{2\Phi}{c^2} \quad (8.85)$$

is dominated by the Newtonian gravitational potential $\Phi = GM/r$. This comparison leads to the following elementary formula for the Schwarzschild radius:

$$r_s = \frac{2GM}{c^2} \quad (8.86)$$
Here are a few examples:

<table>
<thead>
<tr>
<th>Mass</th>
<th>Schwarzschild radius</th>
</tr>
</thead>
<tbody>
<tr>
<td>Electron mass $9.1 \cdot 10^{-31}$ kg</td>
<td>$1.3 \cdot 10^{-60}$ m (far below the Planck length)</td>
</tr>
<tr>
<td>Planck mass $2 \cdot 10^{-8}$ kg</td>
<td>Planck length $1.6 \cdot 10^{-35}$ m</td>
</tr>
<tr>
<td>Human scale 1 kg</td>
<td>$1.5 \cdot 10^{-27}$ m, smaller than the resolution of CERN</td>
</tr>
<tr>
<td>Mass of the Earth $5.9 \cdot 10^{24}$ kg</td>
<td>7 mm</td>
</tr>
<tr>
<td>Mass of the Sun $2.0 \cdot 10^{30}$ kg</td>
<td>3 km</td>
</tr>
<tr>
<td>Mass of the Universe $1.6 \cdot 10^{55}$ kg</td>
<td>$10^{28}$ m $\approx$ visible horizon of the Universe.</td>
</tr>
</tbody>
</table>

Of course, these examples have no concrete meaning, since the Schwarzschild radii are smaller than the considered objects, given the fact that the metric is only valid outside of the objects. They are only intended to convey a vague idea of the orders of magnitude.

### The Flamm paraboloid

As a curved manifold in 3+1 dimensions, the Schwarzschild metric is difficult to imagine. One way to visualize the Schwarzschild metric is to

- select a constant-time slice $t = \text{const}$,
- choose a particular angle for $\theta$, e.g. $\theta = \pi/2$.

Fixing two of the four degrees of freedom, we are left with a two-dimensional manifold with the Schwarzschild metric

$$ds^2 = \left(1 - \frac{r_s}{r}\right)^{-1} dr^2 + r^2 d\phi^2.$$  \hfill (8.87)

In order to visualize this two-dimensional manifold, we embed it into the ordinary three-dimensional $\mathbb{R}^3$ by adding an extra coordinate $z$ and interpreting $r, \phi, z$ as cylindrical coordinates. Such cylindrical coordinates are described by the line element

$$ds^2 = dz^2 + dr^2 + r^2 d\phi^2.$$ \hfill (8.88)

Now suppose that the manifold can be described by a function $z(r)$. This allows us to rewrite line element as

$$ds^2 = \left(1 + \left(\frac{dz(r)}{dr}\right)^2\right) dr^2 + r^2 d\phi^2.$$ \hfill (8.89)

Comparing this expression with Eq. (8.87) we are led to the differential equation

$$z'(r) = \frac{1}{\sqrt{\frac{r}{r_s} - 1}}$$ \hfill (8.90)

with the solution

$$z(r) = \int \frac{1}{\sqrt{\frac{r}{r_s} - 1}} = 2\sqrt{(r - r_s)r_s} + \text{const}.$$ \hfill (8.91)
Towards Gravity

Figure 8.9.: The Flamm paraboloid as a visualization of the Schwarzschild metric.

This can be plotted in Mathematica® via

\[
\text{ParametricPlot3D}\{r \cos[\phi], r \sin[\phi], 2 \sqrt{(r-r_s)rs}\},
\{r,0,5\}, \{\phi,0,2\ \pi\}\}
\]

as shown in Fig. 8.9. As can be seen, the space ‘ends’ at the Schwarzschild horizon vertically with an infinite slope. The interior of the black hole is not accessible and causally disconnected from the exterior.

**Gravitational red shift**

The exterior Schwarzschild metric

\[
ds^2 = -\left(1 - \frac{r_s}{r}\right) dt^2 + \left(1 - \frac{r_s}{r}\right)^{-1} dr^2 + r^2 (d\theta^2 + \sin^2 \theta \ d\phi^2)
\]

exhibits to singularities, namely

\[
\begin{align*}
  r = r_s &: \text{ an apparent singularity at the Schwarzschild radius} \\
  r = 0 &: \text{ a physical singularity in the center at } r = 0
\end{align*}
\]

The apparent singularity is a coordinate-related singularity, much like spherical coordinates at the North Pole of a sphere have singular properties, although in reality there is no singularity at the North Pole. The apparent singularity in the Schwarzschild metric is a consequence of the time coordinate chosen here which correspond to a clock at an infinite distance. As one can read directly from the form of the Schwarzschild metric, clocks in the vicinity of the center are slowed down by the factor \(\sqrt{1 - r_s/\tilde{r}}\) and finally seem to freeze at the Schwarzschild radius.

**Remark:** In their own frame of reference, however, the clocks do not freeze at the Schwarzschild radius, rather they continue to work as usual. The impression of freezing is only an appar-
The gravitational time dilation causes what is called the \textit{gravitational red shift}. In fact, if a source in the vicinity of the Schwarzschild radius emits light at a given frequency $\nu_e$, and observer at infinite distance will perceive a red-shifted frequency $\nu_r = \nu_e \sqrt{1 - r_s/r}$. In astrophysics the \textit{redshift} is quantified by the dimensionless quantity

$$z = \frac{\lambda_r - \lambda_e}{\lambda_e}. \quad (8.92)$$

Using this notation, the gravitational red shift is given by

$$z = \frac{1}{\sqrt{1 - \nu_f/\nu_e}} - 1 \approx \frac{r_s}{2r}. \quad (8.93)$$

For example, the radiation of the Sun perceived on Earth is red-shifted by $z \approx 2 \cdot 10^{-6}$, corresponding to a time difference between the corresponding clocks at the surface of the sun and on earth by 19 hours per 1000 years.
List of frequently used abbreviations

SR  Special Relativity
GR  General Relativity
Appendices.

A. Sign conventions

In these lecture notes we work with the ‘mostly plus’ metric

\[ \eta_{\mu\nu} = \eta^{\mu\nu} = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \] (A.1)

Depending on this choice we use the following definition.

- The invariant line element along a timelike trajectory reads
  \[ ds = \sqrt{-dx_\mu dx^\mu} = \sqrt{-\dot{x}_\mu \dot{x}^\mu} \, d\tau. \] (A.2)

- The d’Alembert operator (quabla) is defined by
  \[ \Box := \partial^\mu \partial_\mu = \partial_\mu \partial^\mu = \nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2}. \] (A.3)

- The Klein-Gordon equation reads
  \[ (\Box - M^2)\phi = 0, \] (A.4)

where \( M = \frac{mc}{\hbar} \) is the mass parameter.
B. Unit systems

This lecture on Special Relativity deals in many places with electrodynamics. In Physics, there are many competing unit systems which create a lot of confusion. In these lecture notes we stick to the internationally accepted systems of SI units. But many other textbooks use Gaussian cgs units. Even more confusing, there are four different kinds of cgs units. This competes with the general attitude of theoretical physicists to set all constants such as the speed of light equal to 1.

Unnecessary units and unnecessary conversion factors

There is of course a deeper reason why Gaussian units are still popular among physicists. The reason is that the SI system introduces a new unit for the electric current, called Ampere (A), or equivalently, Coulomb (C=As) for the electric charge. This definition is a historical one, but it is in fact not really needed. The reason is that the unit of the electric current can be expressed in the elementary units of length, time, and mass. In fact, it would have been much more meaningful to define an Ampere as one \( m^{3/2} kg^{1/2} s^{-2} \).

Let us imagine a similar situation with the unit of forces. One Newton is defined in terms of the elementary units as 1N=kg m/s\(^2\). Of course, we could have defined one Newton as 0.7257 kg m/s\(^2\), but then it is clear that the crazy factor 0.7257 would appear everywhere in the formulas. We would perhaps call this factor \( \nu_0 = 0.7257 \), and our formulas would be decorated with \( \nu_0 \)'s almost everywhere. Hence, for introducing an irrelevant arbitrary unit, we would have to pay the price of including an irrelevant conversion factor.

This is exactly what happened in electrodynamics in the SI system. The unit Ampere was defined historically in an arbitrary way. For this we have to pay the price of introducing an unnecessary conversion factor, the electric permittivity in vacuum

\[
\epsilon_0 = 8.854\;187\;8176\ldots \times 10^{-12}\;\text{As}\;\text{Vm}. \tag{B.5}
\]

In addition, one defines the vacuum permeability as \( \mu_0 = \frac{1}{\epsilon_0 c^2} \), which is essentially the reciprocal of \( \epsilon_0 \). And in fact, if you look at the equations of electrodynamics in SI units, they are decorated in many places with \( \epsilon_0 \)'s and \( \mu_0 \)'s.

A similar story happened in thermodynamics. The experimentalists learned to measure temperature, and they defined arbitrary units for it, namely, Celsius, Fahrenheit, or Kelvin. They did not know that temperature is energy per bit of information, so we could simply use the same unit as for energy, namely m\(^2\)kg/s\(^2\). But historically a different unit was defined and it is now part of SI units. The price we pay is an irrelevant conversion factor, known as the Boltzmann constant \( k_B \approx 1.38065 \times 10^{-23}\;\text{m}^2\;\text{kg}\;\text{s}^{-2}\;\text{K}^{-1} \).

From Gauß units to SI units and back

So, the SI system has clear disadvantages in that it carries unnecessary units and unnecessary...
necessary conversion factors. However, it is the internationally accepted system of units, all technical applications are based on the SI system. Converting results from Gaussian units or other cgs variants to SI units is awkward and highly error-prone. As an example let us mention the minimal coupling of an electromagnetic field to the dynamics of a point particle, as expressed by the replacement

\[
\text{SI-system: } \mathbf{p} \rightarrow \mathbf{p} - e \mathbf{A}, \quad \text{Gauss system: } \mathbf{p} \rightarrow \mathbf{p} - \frac{e}{c} \mathbf{A} \quad (B.6)
\]

A comprehensive summary for converting Gauß to SI units an back can be found in

[https://en.wikipedia.org/wiki/Gaussian_units](https://en.wikipedia.org/wiki/Gaussian_units)

Here are the conversion rules for some key quantities:

<table>
<thead>
<tr>
<th>quantity</th>
<th>symbol</th>
<th>SI unit</th>
<th>factor</th>
<th>Gauss unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>space</td>
<td>(\bar{x}, x)</td>
<td>m</td>
<td>100</td>
<td>cm</td>
</tr>
<tr>
<td>time</td>
<td>(t)</td>
<td>s</td>
<td>1</td>
<td>s</td>
</tr>
<tr>
<td>mass</td>
<td>(m)</td>
<td>kg</td>
<td>1000</td>
<td>g</td>
</tr>
<tr>
<td><strong>Elementary quantities</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>energy, work</td>
<td>(E, W)</td>
<td>J = kg m^2 s^{-2}</td>
<td>10^7</td>
<td>erg = g cm^2 s^{-2}</td>
</tr>
<tr>
<td>momentum</td>
<td>(\vec{p}, p)</td>
<td>kg m s^{-1}</td>
<td>10^5</td>
<td>g cm s^{-1}</td>
</tr>
<tr>
<td>force</td>
<td>(\vec{F})</td>
<td>N = kg m s^{-2}</td>
<td>10^5</td>
<td>dyne = g cm s^{-2}</td>
</tr>
<tr>
<td>action</td>
<td>(S)</td>
<td>Js = kg m^2 s^{-1}</td>
<td>10^7</td>
<td>erg s = g cm^2 s^{-1}</td>
</tr>
<tr>
<td><strong>Mechanics</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>electric current</td>
<td>(I)</td>
<td>A</td>
<td></td>
<td></td>
</tr>
<tr>
<td>electric charge</td>
<td>(q_e)</td>
<td>C=A s</td>
<td></td>
<td></td>
</tr>
<tr>
<td>electric potential</td>
<td>(U, \phi)</td>
<td>V = kg m^2 A^{-1} s^{-3}</td>
<td></td>
<td></td>
</tr>
<tr>
<td>electric field</td>
<td>(\vec{E})</td>
<td>V/m = kg m A^{-1} s^{-3}</td>
<td></td>
<td></td>
</tr>
<tr>
<td>vector potential</td>
<td>(\vec{A})</td>
<td>T m=kg m A^{-1} s^{-2}</td>
<td></td>
<td></td>
</tr>
<tr>
<td>magnetic field</td>
<td>(\vec{B})</td>
<td>T=kg A^{-1} s^{-2}</td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Electrodynamics</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Here are a few important equations of electrodynamics in both unit systems:

<table>
<thead>
<tr>
<th>Relation</th>
<th>SI units</th>
<th>Gauss units</th>
</tr>
</thead>
<tbody>
<tr>
<td>4-potential</td>
<td>$A^\mu = (\frac{1}{c}\phi, \vec{A})$</td>
<td>$A^\mu = (\phi, \vec{A})$</td>
</tr>
<tr>
<td>electric field</td>
<td>$\vec{E} = -\nabla \phi - \partial_t \vec{A}$</td>
<td>$\vec{E} = -\nabla \phi - \frac{1}{c} \partial_t \vec{A}$</td>
</tr>
<tr>
<td>magnetic field</td>
<td>$\vec{B} = \nabla \times \vec{A}$</td>
<td>$\vec{B} = \nabla \times \vec{A}$</td>
</tr>
<tr>
<td>Lorentz gauge $\partial_\mu A^\mu = 0$</td>
<td>$\frac{1}{c} \partial_t \phi + \nabla \cdot \vec{A} = 0$</td>
<td>$\frac{1}{c} \partial_t \phi + \nabla \cdot \vec{A} = 0$</td>
</tr>
<tr>
<td>Inh. Maxwell equations</td>
<td>$\Box A^\mu = -\mu_0 J^\mu$</td>
<td>$\Box A^\mu = -\frac{4\pi}{c} J^\mu$</td>
</tr>
</tbody>
</table>
Figure C.1.: Illustration of a Legendre transformation. Left: Convex function \( f(x) = \cosh(x) \). Middle: Suppose our aim is to determine the value of \( f^* \) at \( m = 2 \). To do this we draw a line \( y = mx \) with the slope \( m = 2 \). Then maximal distance between this straight line and the function \( f(x) \), marked by the red vertical line, is the value of the Legendre transformation for this particular slope. Alternatively, we may draw \( y = mx - f(x) \) and search for the maximum. Right: If we do this for all \( m \) we obtain the Legendre transform \( f^*(m) \), which is also a convex function.

C. Legendre transformations

For converting a Lagrange function into a Hamiltonian we need a Legendre transformation. In order to understand this transformation, let us first discuss its mathematical definition and properties.

Starting point is a function \( f(x) \). Plotting this function, we usually parameterize the points of the curve by the independent variable \( x \). The basic idea behind a Legendre transformation is to replace the parameter \( x \) by the slope of the curve \( m = f'(x) \), that is, we would like to parameterize the curve by \( m \) instead of \( x \). Of course, this is only possible if there is a unique invertible relation \( x \leftrightarrow m \), and obviously this requires \( f(x) \) to be either strictly convex or strictly concave.

Mathematically a Legendre transformation is defined as follows: Let \( f(x) \) be a convex function defined on the interval \( I \subseteq \mathbb{R} \). Then the convex conjugate function, better known as Legendre transform \( f^*(m) \) is defined by

\[
f^*(m) := \sup_{x \in I} \left( mx - f(x) \right).
\]  

This definition is valid for all \( m \in I^* \) were the supremum exists. In Fig. C.1 we illustrate how the supremum can be computed.

The most important properties of the Legendre transformation are

- The Legendre transform \( f^*(m) \) is also a convex function.
- As can be shown easily, the Legendre transformation is an involution, i.e., it is identical with its own inverse: \( f^{**}(x) = f(x) \), or in short: \( f^{**} = f \).

Usually definitions in terms of the supremum are difficult to realize in practice because one has to search for the maximum which consumes a lot of computer time. However, in cases where the original function is differentiable, we can locate the maximum analytically which allows us to compute the Legendre transform in a closed form. In this case, for given \( m \), the supremum is exactly located at the position \( x \) where the derivative vanishes:

\[
\frac{d}{dx} \left( mx - f(x) \right) = m - f'(x) = 0 \quad \Rightarrow \quad m = f'(x).
\]
This means that \( m \) is just the slope of the function \( f \) evaluated at this particular position. Now the only non-trivial step in determining the Legendre transformation is to solve the equation \( m = f'(x) \) for \( x \). The solution is of the form \( x = g(m) \), where \( g \) is just the inverse function of the derivative \( f' \). Having computed the inverse function \( g \), we can now write down the Legendre transform explicitly:

\[
f^*(m) = mg(m) - f(g(m)).
\] (C.9)

As can be checked easily, the derivative of the Legendre transform is nothing but the function \( g \) itself:

\[
g(m) = f'^*(m).
\] (C.10)

**Example:** In order to determine the Legendre transform of the convex function \( f(x) = \cosh(x) \) we first compute the derivative \( m = f'(x) = \sinh(x) \). This equation can be inverted in a closed form as \( x = g(m) = \text{arcsinh}(m) \). This allows one to compute the Legendre transforming explicitly:

\[
f^*(m) = m \text{arcsinh}(m) - \cosh(\text{arcsinh}(m))
= m \text{arcsinh}(m) - \sqrt{1 - m^2}.
\] (C.11)

Note that this function is convex, too. The value of this function at \( m = 2 \) is \( f^*(2) \approx 0.651 \), in agreement with example shown in Fig. C.1.

**Legendre transformation of concave functions**

For concave functions, the Legendre transform is simply given by the infimum

\[
f^*(m) := \inf_{x \in I} \left( mx - f(x) \right).
\] (C.12)

Apart from that everything else remains the same, i.e., we set \( m = f'(x) \), compute the inverse \( x = g(m) \), and finally get \( f^*(m) = mg(m) - f(g(m)) \). This function is concave as well.

**Example:** In order to determine the Legendre transform of the concave function \( f(x) = \ln(x) \) we compute the derivative \( m = f'(x) = 1/x \), hence \( x = g(m) = 1/m \). The resulting Legendre transform is also a concave function:

\[
f^*(m) = 1 + \ln m
\] (C.13)
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